TRIANGULAR ICE: COMBINATORICS & LIMIT SHAPES

(PDF + E. Guitter IPHT Saclay)

+ B. Debin UC Louvain

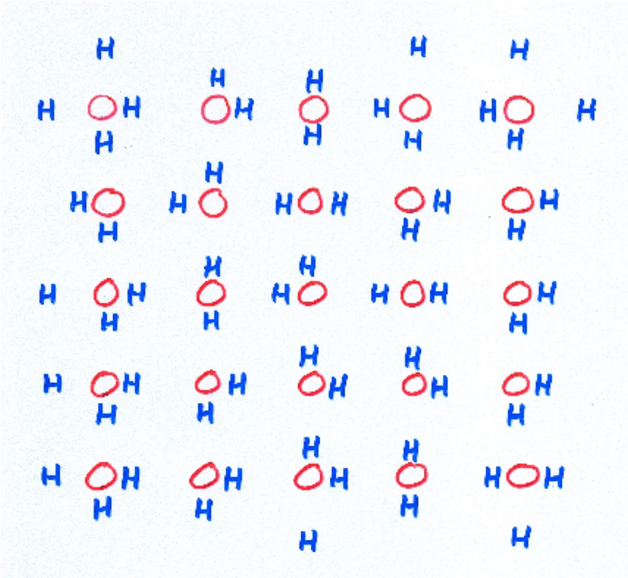
1. ASM, square ice, 6V model and integrability
2. Triangular ice, DWBC
3. Domino Tilings of the Holey Aztec Square
4. Proof of the APM – HASDT correspondence
5. Combinatorial Conjectures
6. Limit shape / Arctic Phenomenon
7. Conclusion

A Tale of **3** sequences

$$\begin{cases} 1, 3, 23, 433, 19705, 2151843, \ldots \\ 1, 3, 29, 901, 89893, 28793575, \ldots \\ 1, 4, 60, 3328, 678912, 508035072, \ldots \end{cases}$$

Varna 22

ASM and Square Ice



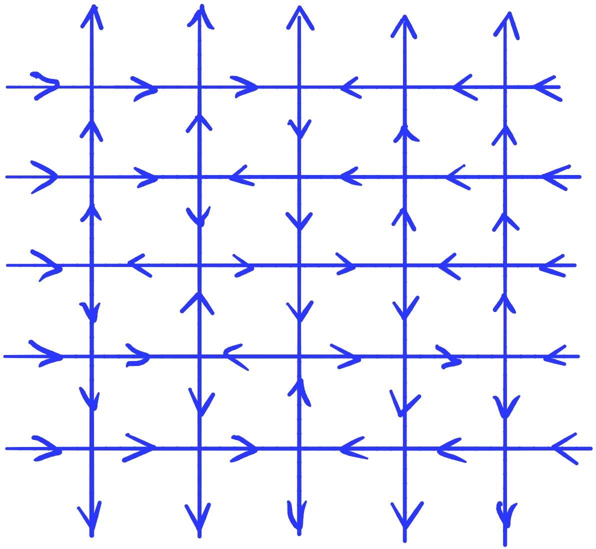
Replace data by dipolar momenta $\{\rightarrow, \leftarrow, \downarrow, \uparrow\}$

Ice Rule at each vertex

incoming arrows

= # outgoing arrows $\Rightarrow$ 6V

ASM and 6V model



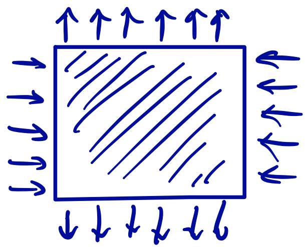
Replace data by dipolar momenta $\{\rightarrow, \leftarrow, \downarrow, \uparrow\}$

Ice Rule at each vertex

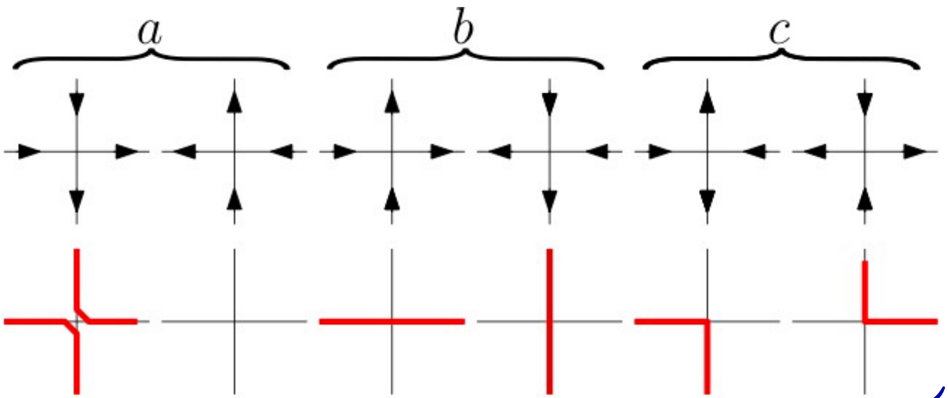
incoming arrows = # outgoing arrows $\Rightarrow$ 6V

+ Domain Wall Boundary Conditions

($n \times n$ square)



Bijections



① 6V configs

② osculating paths

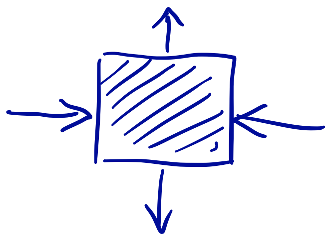
(NW → SE)

Transmittor vertices | Reflector vertices

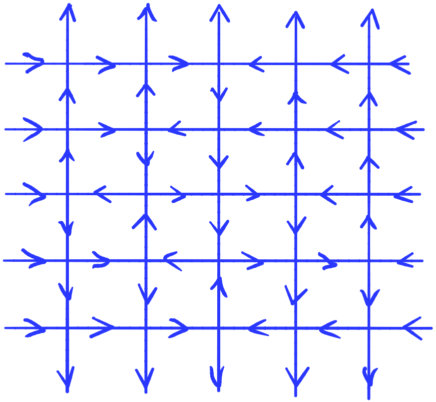
0	0	0	0	+1	−1

③ ASM entries

Alternance conditions

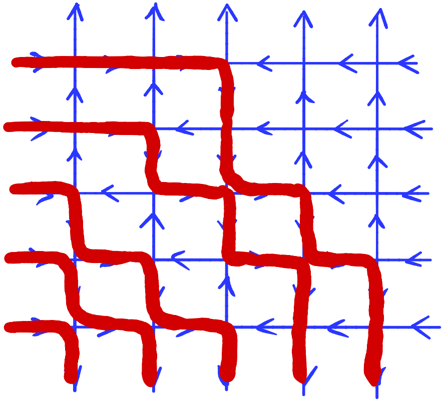


odd # of reflections!



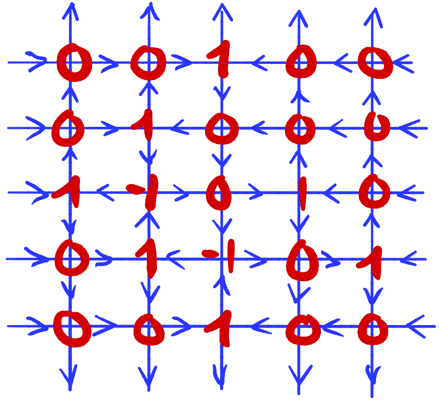
6V

1



Osculating Paths

2



ASM

3

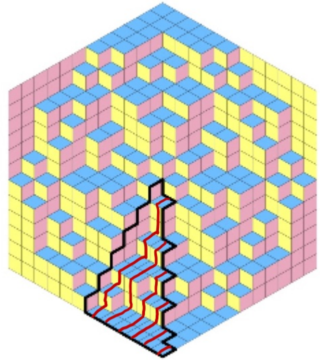
n

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

n

ASM

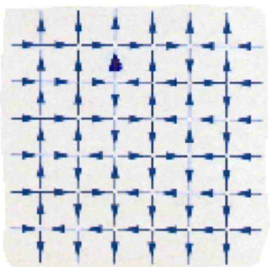
$$ASM_n = \frac{\prod_{i=0}^{n-1}(3i+1)!}{\prod_{i=0}^{n-1}(n+i)!}$$



2n

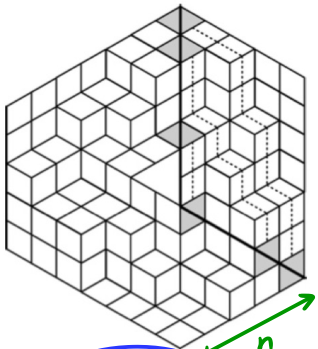
TSSC PP

n



n

6V DWBC



n+2

n

$\circlearrowleft \frac{2\pi}{3}$

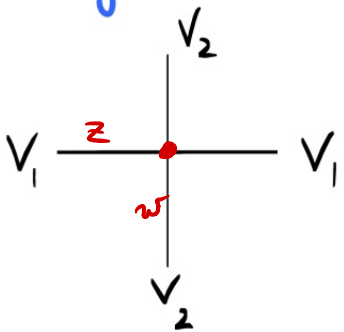
DPP

Korepin, Izergin
Andrews
Kuperberg, Zeilberger
Razumov, Stroganov
Cantini Sportiello
Zinn-Justin, P.D.F., Behrend
DeGier, Nienhuis
Knutson, Krattenthaler, Fisher

INTEGRABILITY

- Boltzmann weights

R operator



$$V_1 \otimes V_2 \xrightarrow{R(z,w)} V_2 \otimes V_1$$

$\dim V_i = 2$

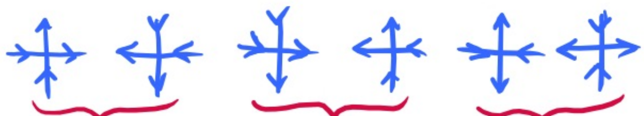
$V_1 = \langle \rightarrow, \leftarrow \rangle \ni \alpha$

$V_2 = \langle \uparrow, \downarrow \rangle \ni \beta$

<u>matrix entries</u> in $\alpha \otimes \beta \to \beta \otimes \alpha$

6 non-zero entries out of 16

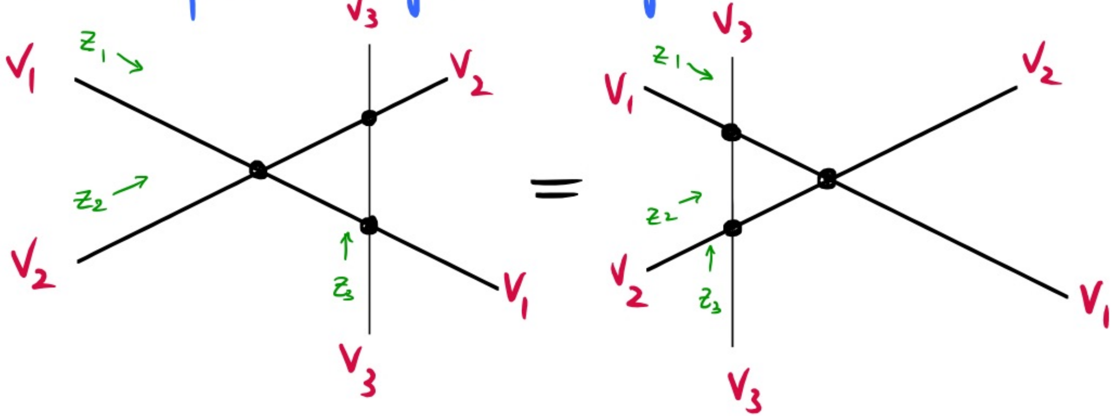
(ice rule).



a b c

YANG-BAXTER RELATION

One can pick "integrable weights" such that:



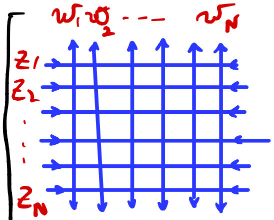
(a cubic identity for R operators from $V_1 \otimes V_2 \otimes V_3$ to $V_3 \otimes V_2 \otimes V_1$

$$\boxed{a(z,w) = z - w \; ; \; b(z,w) = q^{-2}z - q^{2}w \; ; \; c(z,w) = (q^{2}-q^{-2})\sqrt{zw}}$$

$\dim V = 2$; TRIGONOMETRIC R-matrix of 6V MODEL

IZERGIN-KOREPIN Determinant

$Z_{6V}\Big[$

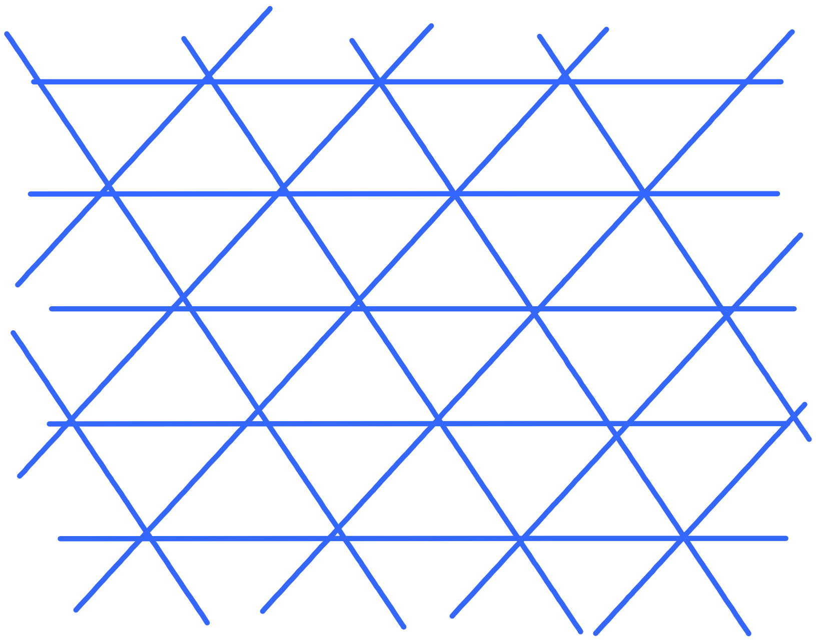


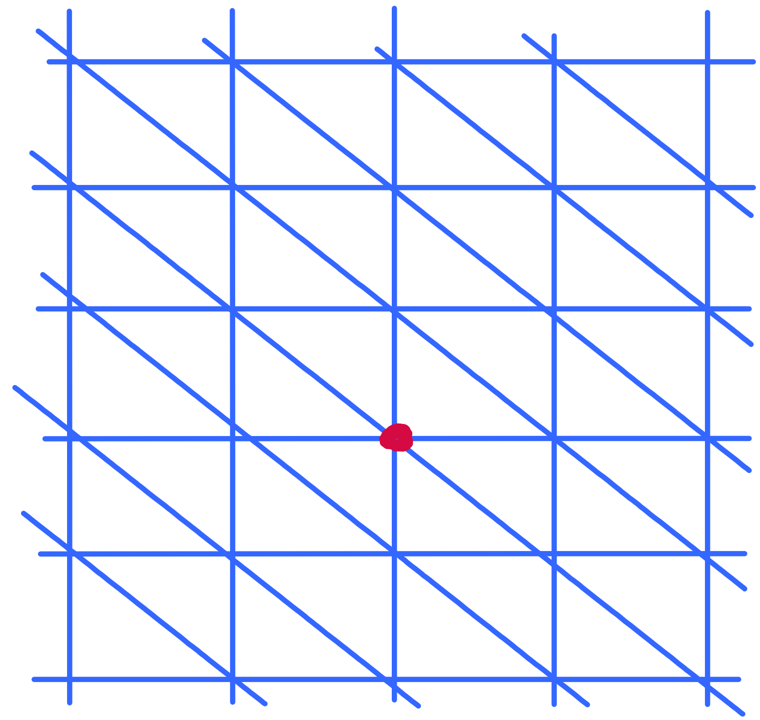
$\Big]$

$$= \frac{\prod_{i,j=1}^{N} a(z_i,w_j)\, b(z_i,w_j)}{\prod_{1\le i<j\le N} (z_i-z_j)(w_i-w_j)} \times$$

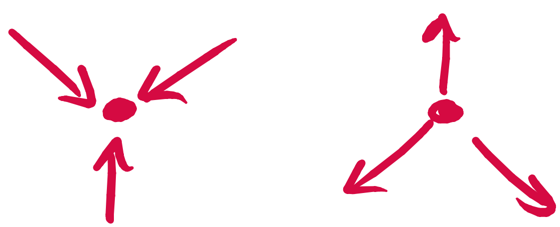
$$\times \det\left(\frac{c(z_i,w_j)}{a(z_i,w_j)\, b(z_i,w_j)}\right)_{1\le i,j\le N}$$

2. TRIANGULAR ICE (20V model)





ice rule



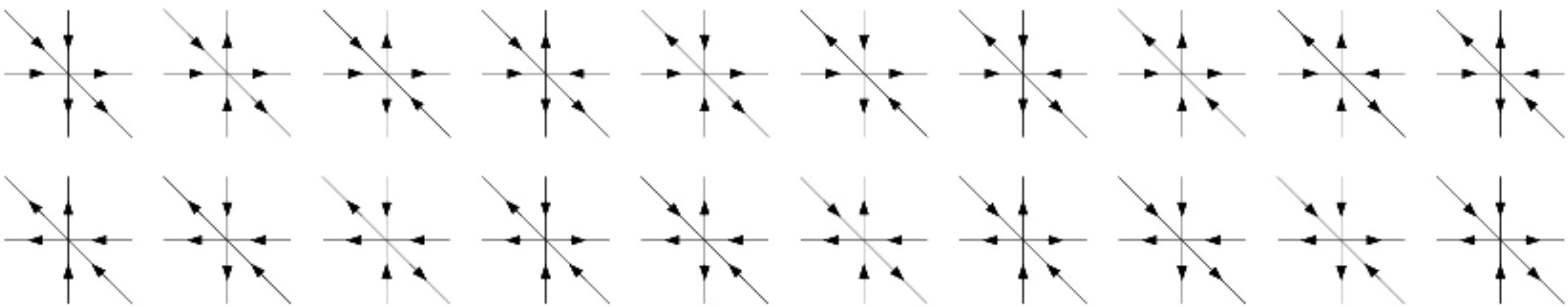
at each vertex

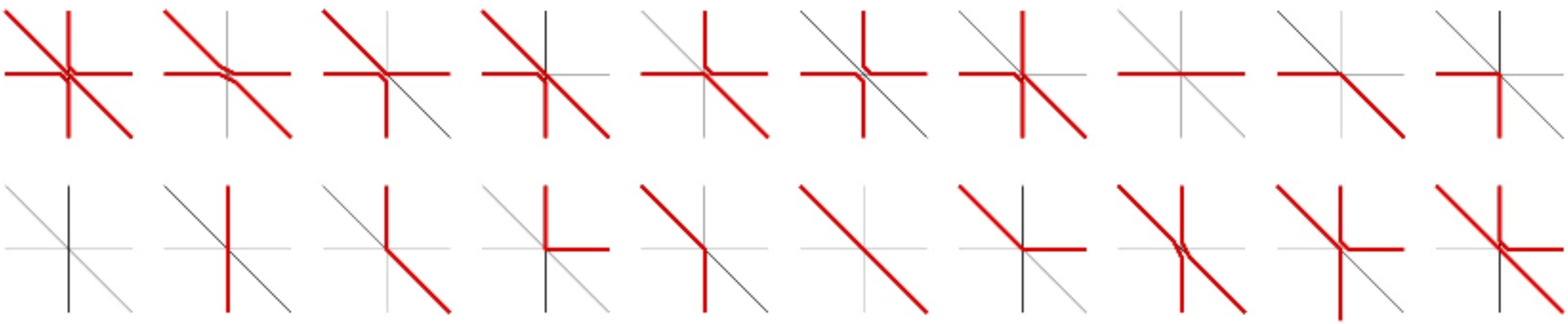
[Kelland, Baxter]

$$\binom{6}{3} = 20$$

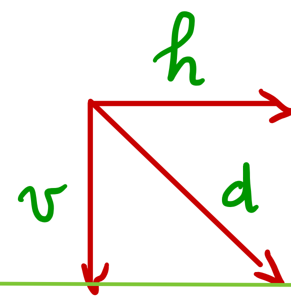
TRIANGULAR ICE (20V model)

Twenty vertices:





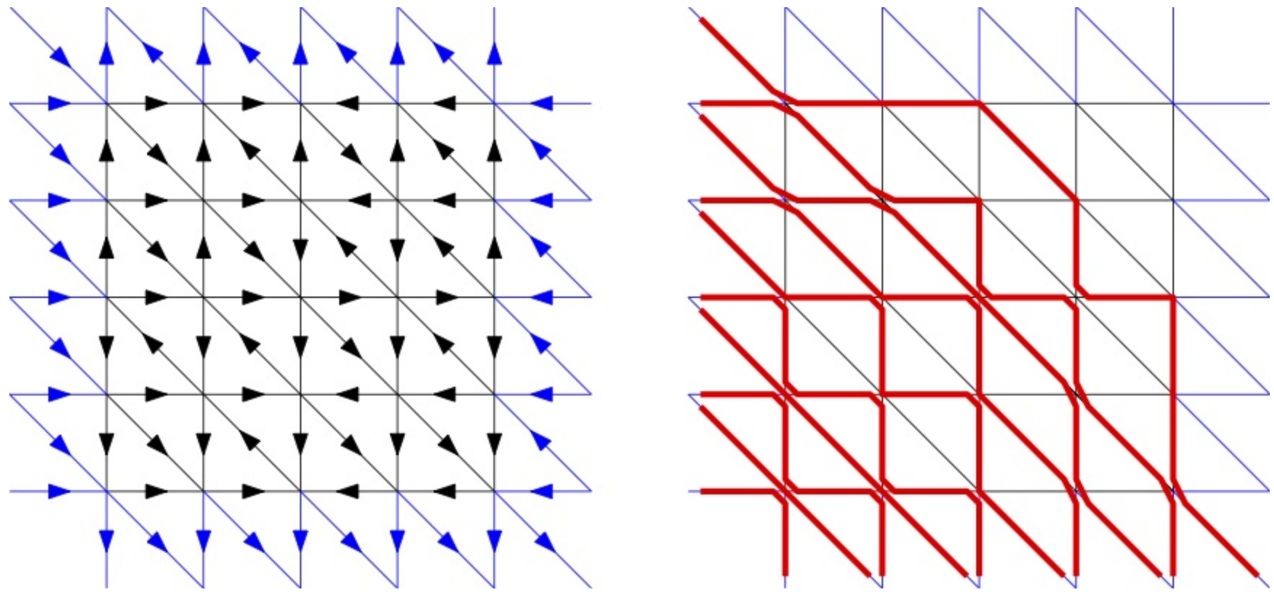
Osculating Schröder paths



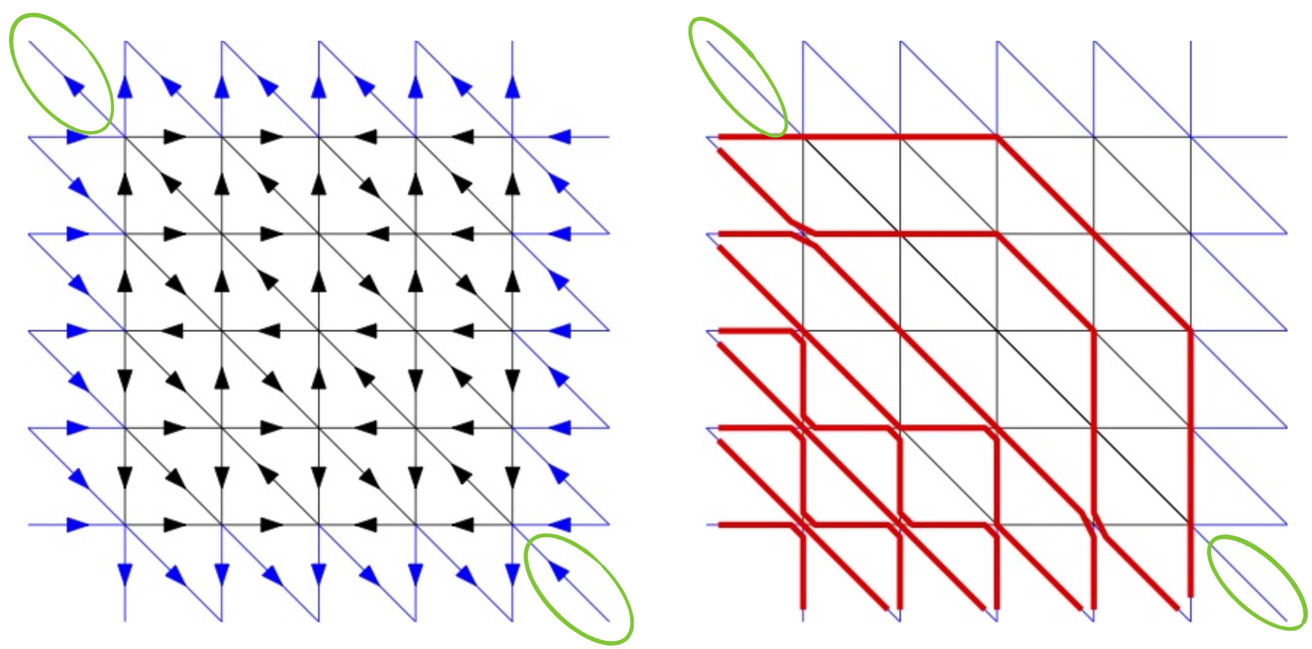
h, v, d steps

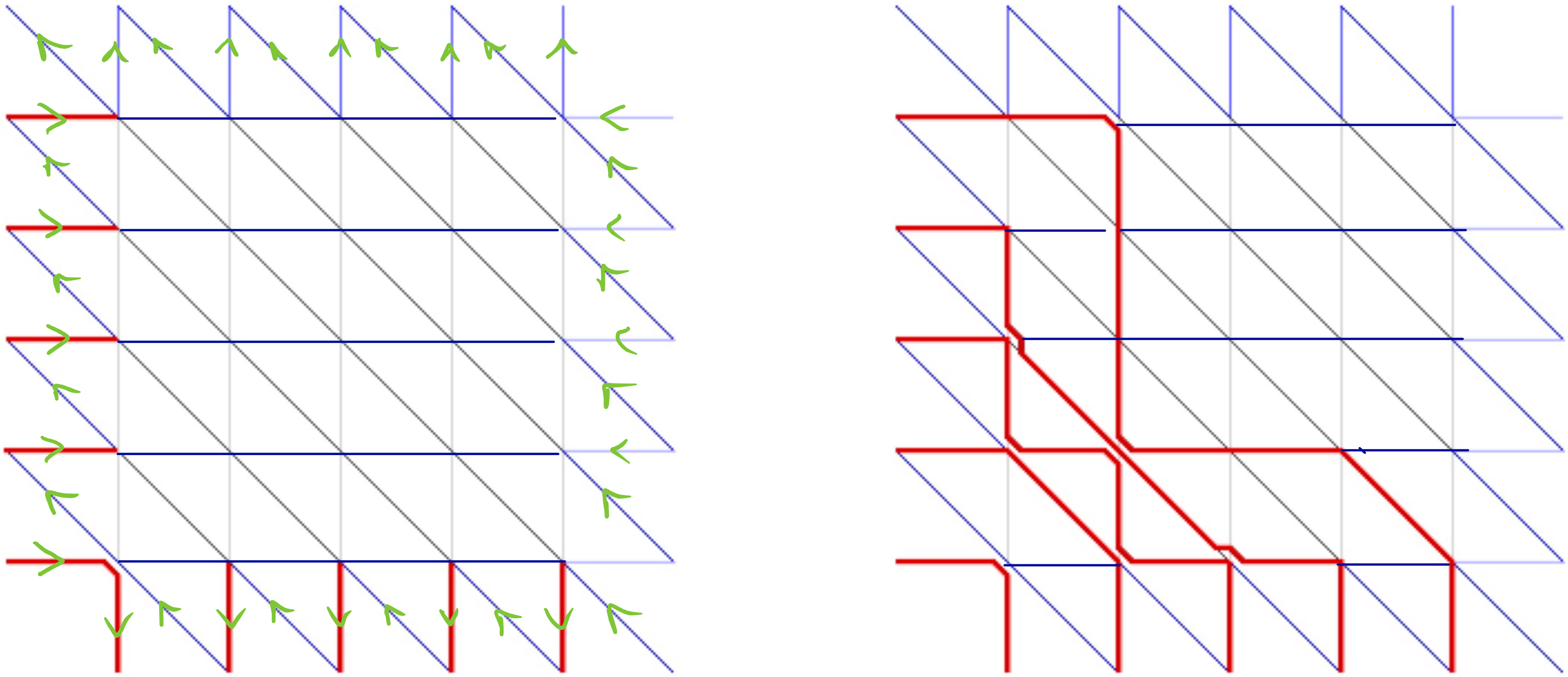
DOMAIN WALL BOUNDARY CONDITIONS

DWBC1



DWBC2





DWBC 3

Numbers of Configurations on an $n \times n$ grid:

DWBC 1,2

$$A_n = 1, 3, 23, 433, 19705, 2151843, \ldots$$

DWBC 3

$$B_n = 1, 3, 29, 901, 89893, 28793575, \ldots$$

[computed by transfer matrix]

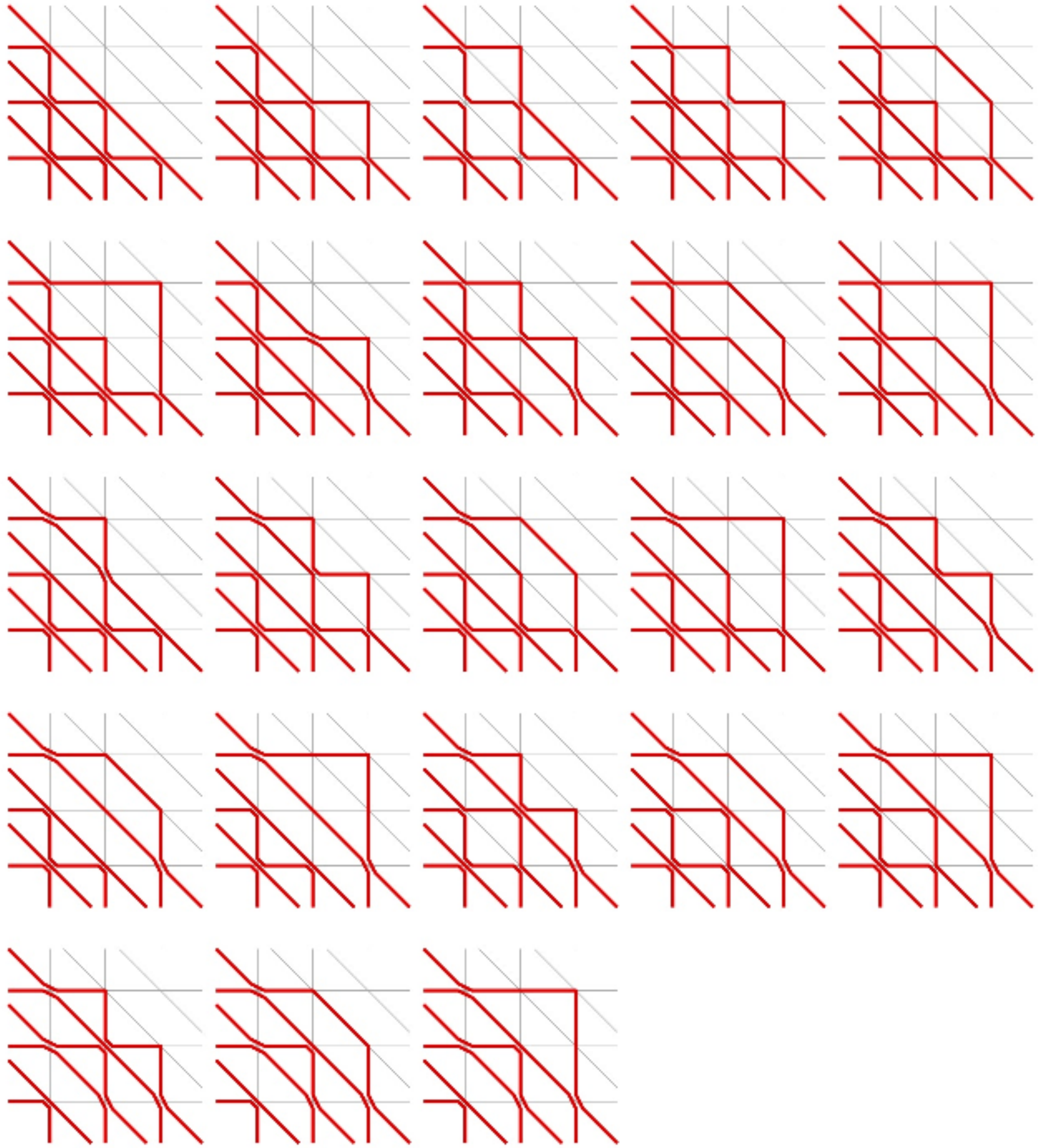
20 v

DWBC 1

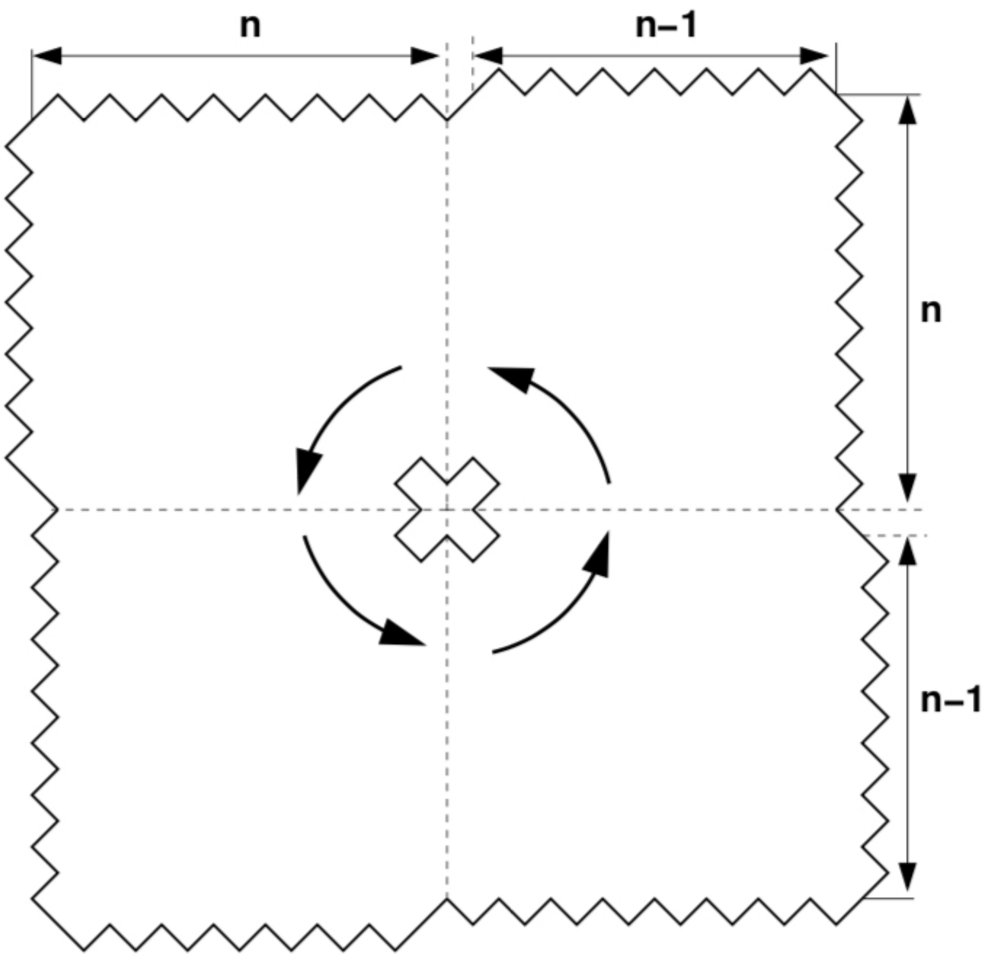
configurations

$n=3$

(23)

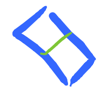
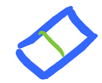


3. DOMINO TILINGS OF THE HOLEY AZTEC SQUARE WITH QUARTER-TURN SYMMETRY



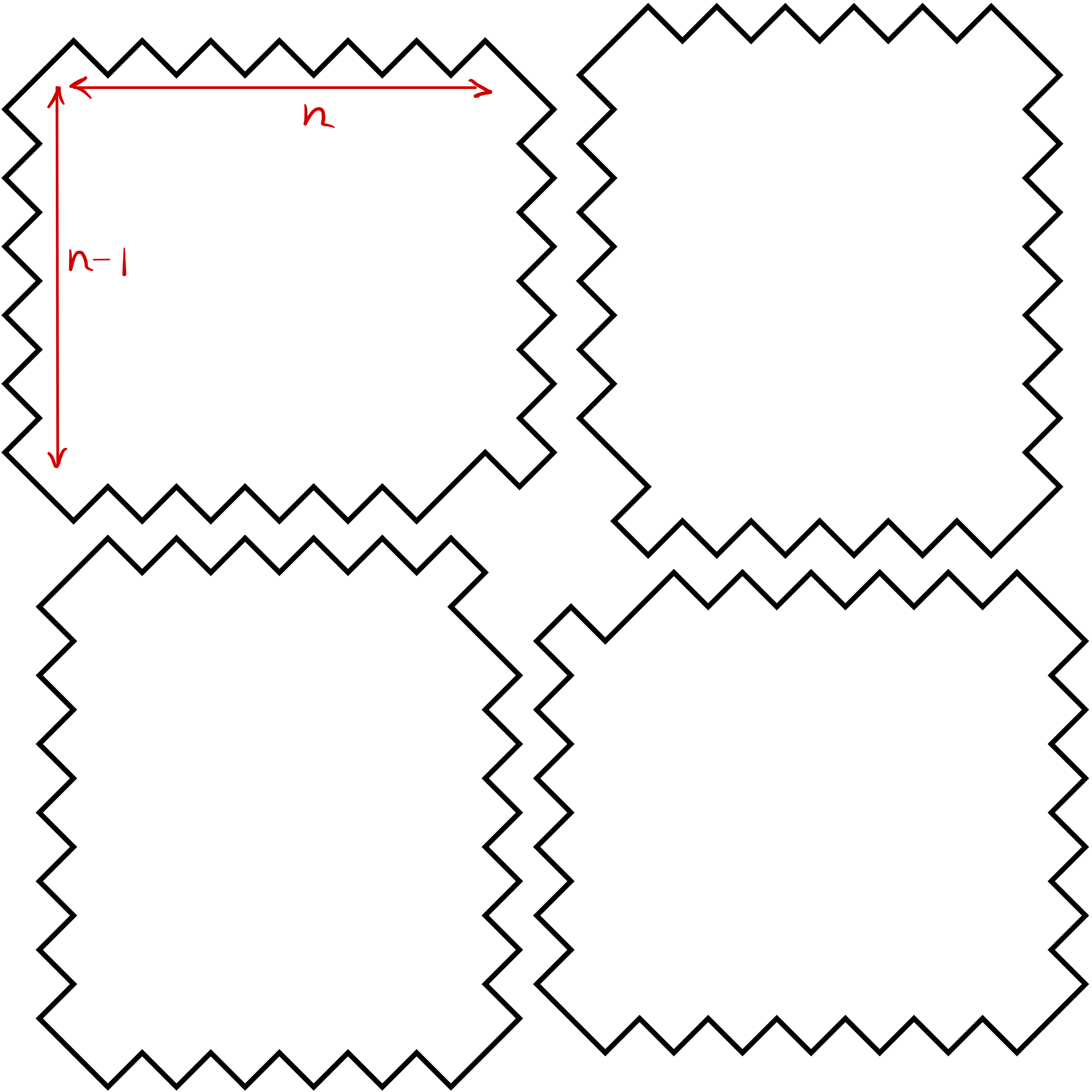
$\mathcal{S}_n$

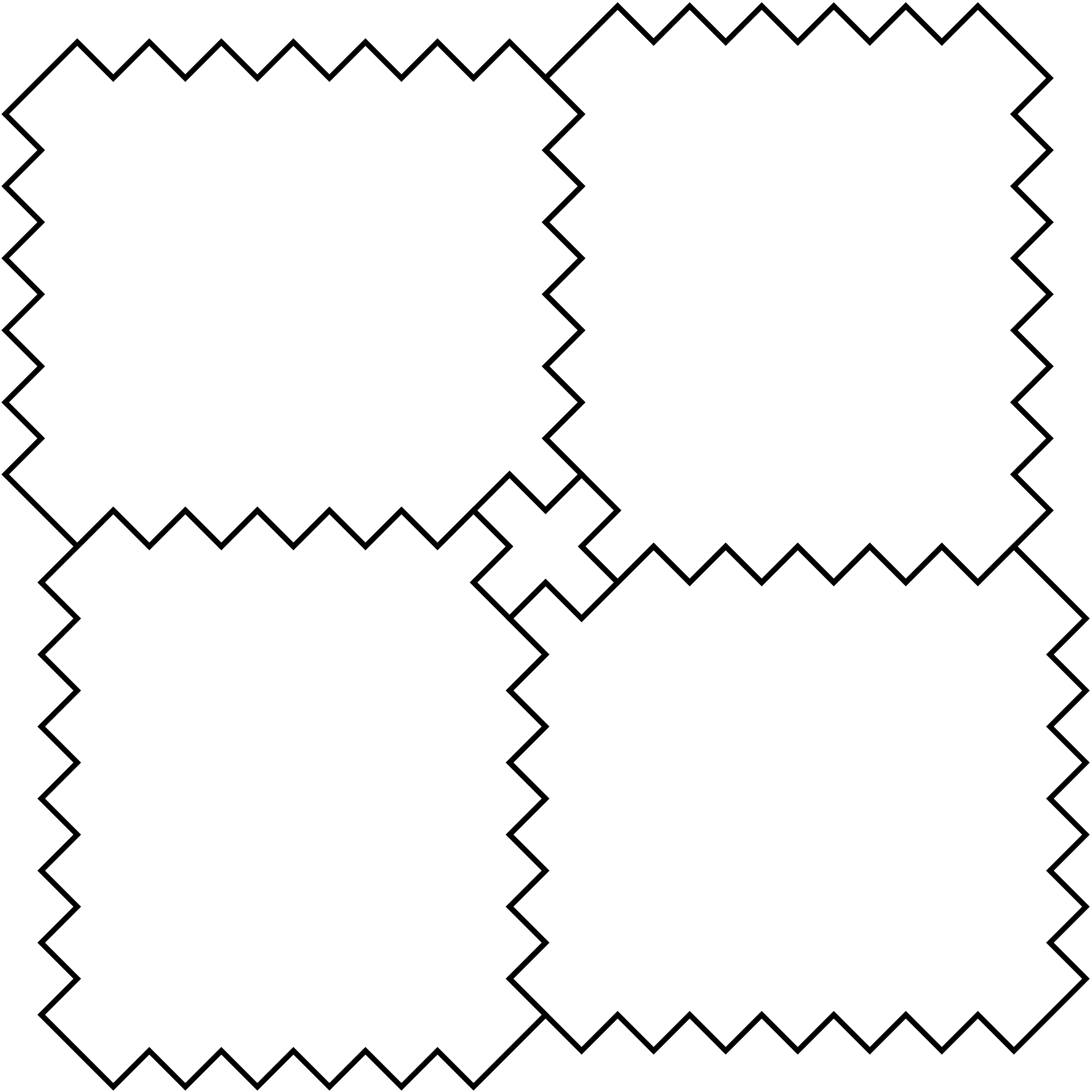
Domino Tilings: use

 and  2×1 dominos

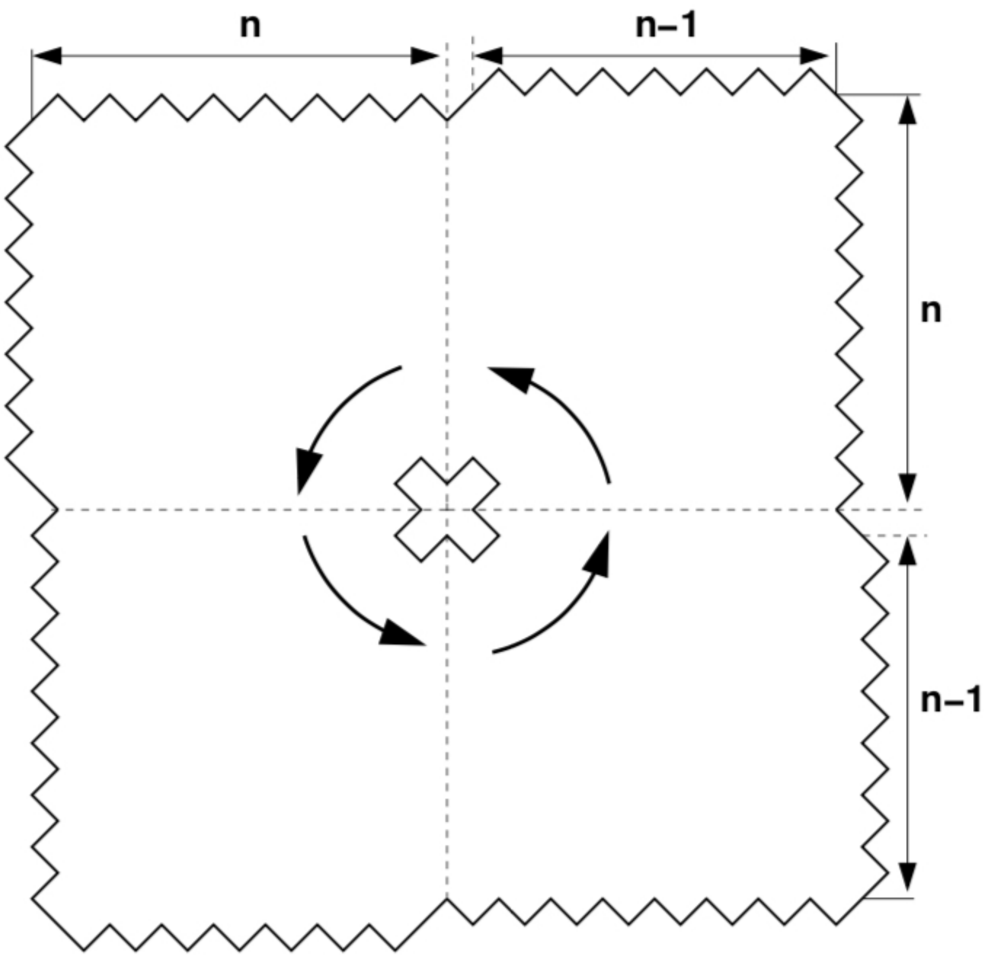
Rotational symmetry by $\frac{\pi}{2}$

NB: the hole makes it tileable!

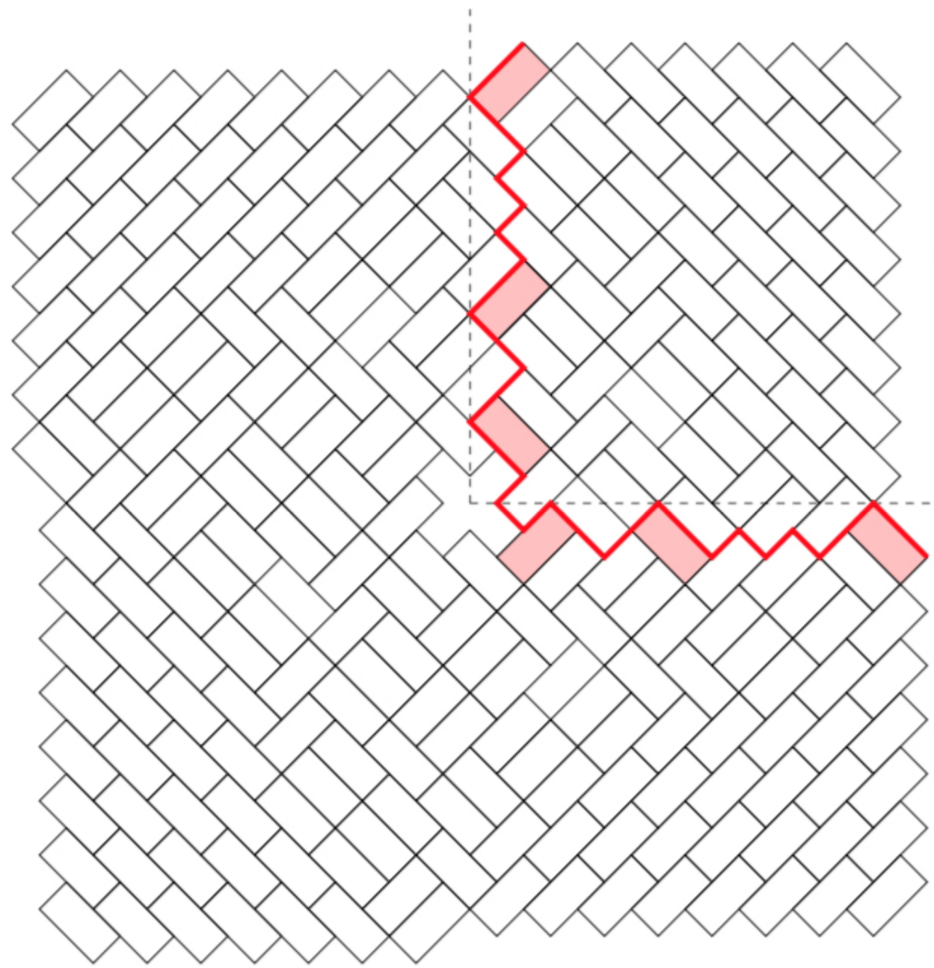

n
n-1



DOMINO TILINGS OF THE HOLEY SQUARE WITH QUARTER-TURN SYMMETRY

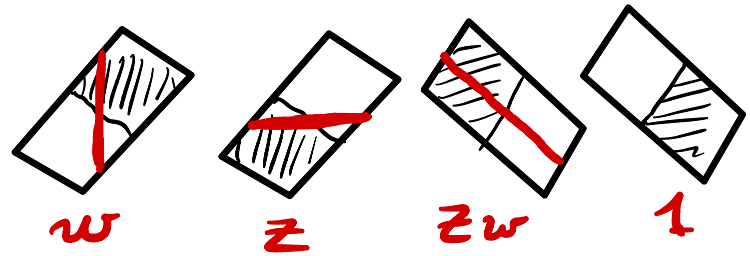


$\mathcal{S}_n$



sample tiling

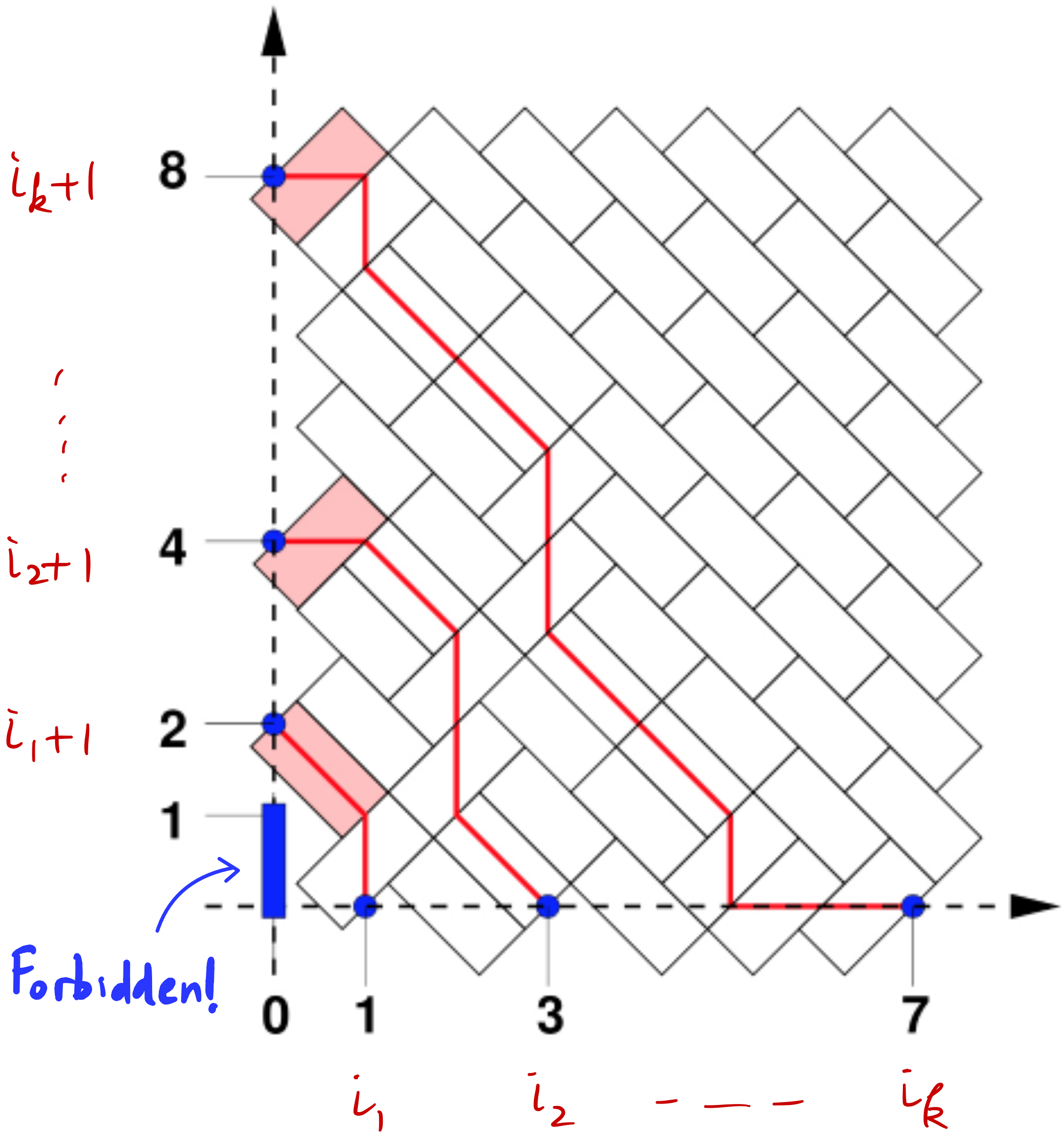
Counting Configurations

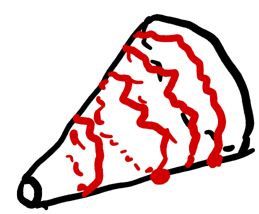


- Non-intersecting Schröder paths w fixed ends

- first step cannot be |

- start and ends identified (cone).





Counting Configurations

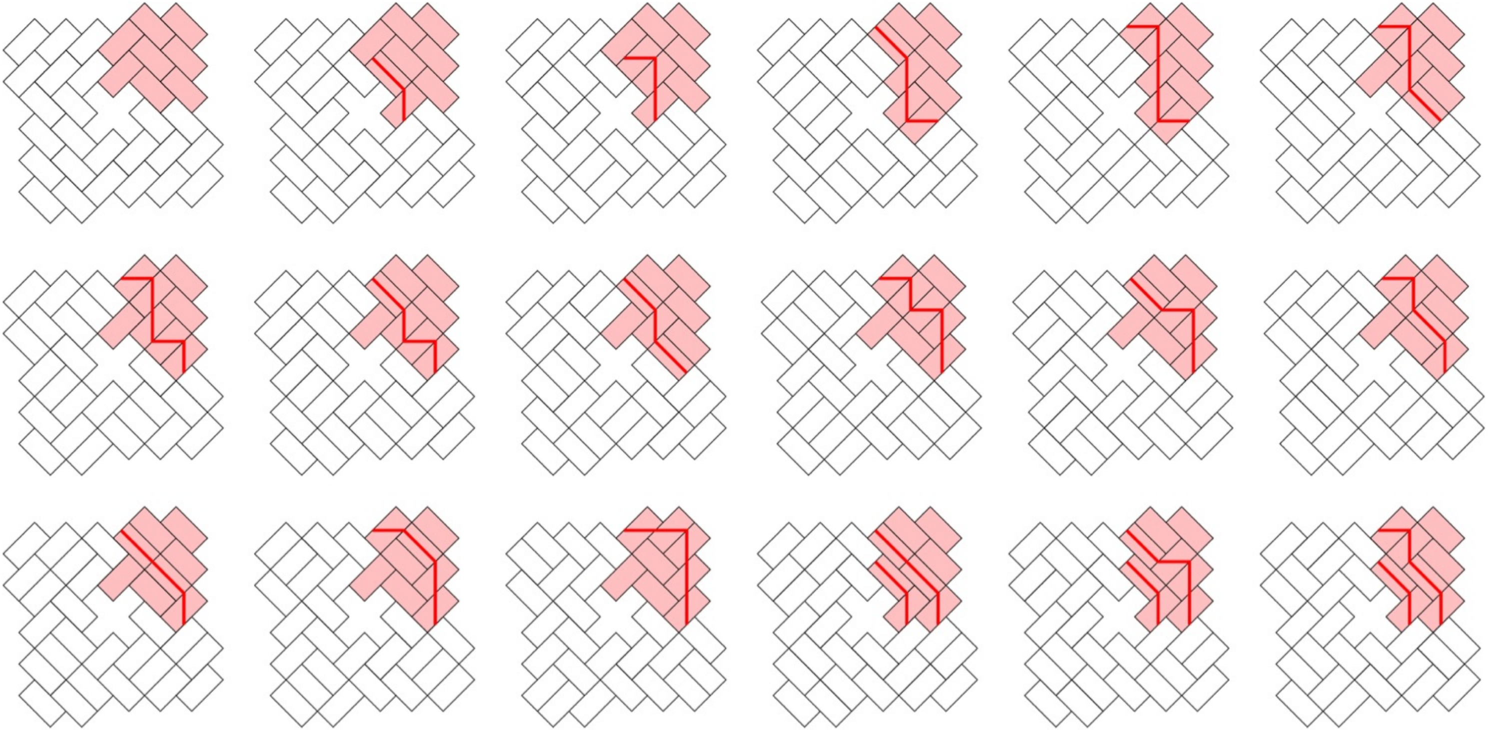
Thm [PDF-Guitter 19]

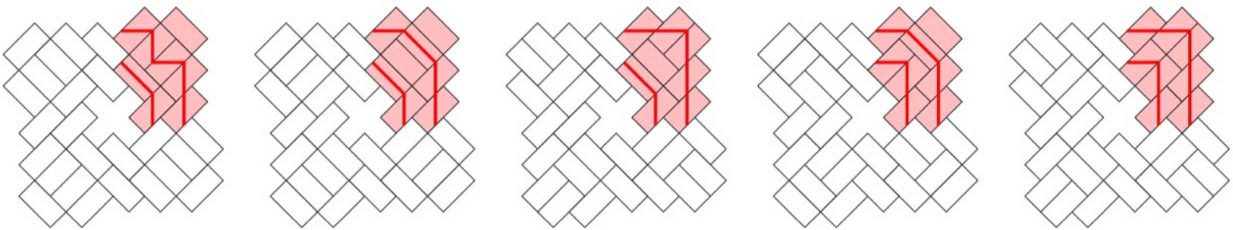
$$T_4(\mathcal{J}_n) = \det_{0\le i,j\le n-1}\left(\left\{ \frac{1}{1-zw} + \frac{2z}{(1-z)(1-z-w-zw)} \right\}_{z^i w^j} \right)$$

Proof: (Cauchy-Binet) $\det(\mathrm{Id} + M) = \sum_{i_1<\dots<i_k} \underbrace{\left| M^{i_1\dots i_k}_{i_1\dots i_k} \right|}$ (Gessel-Viennot)

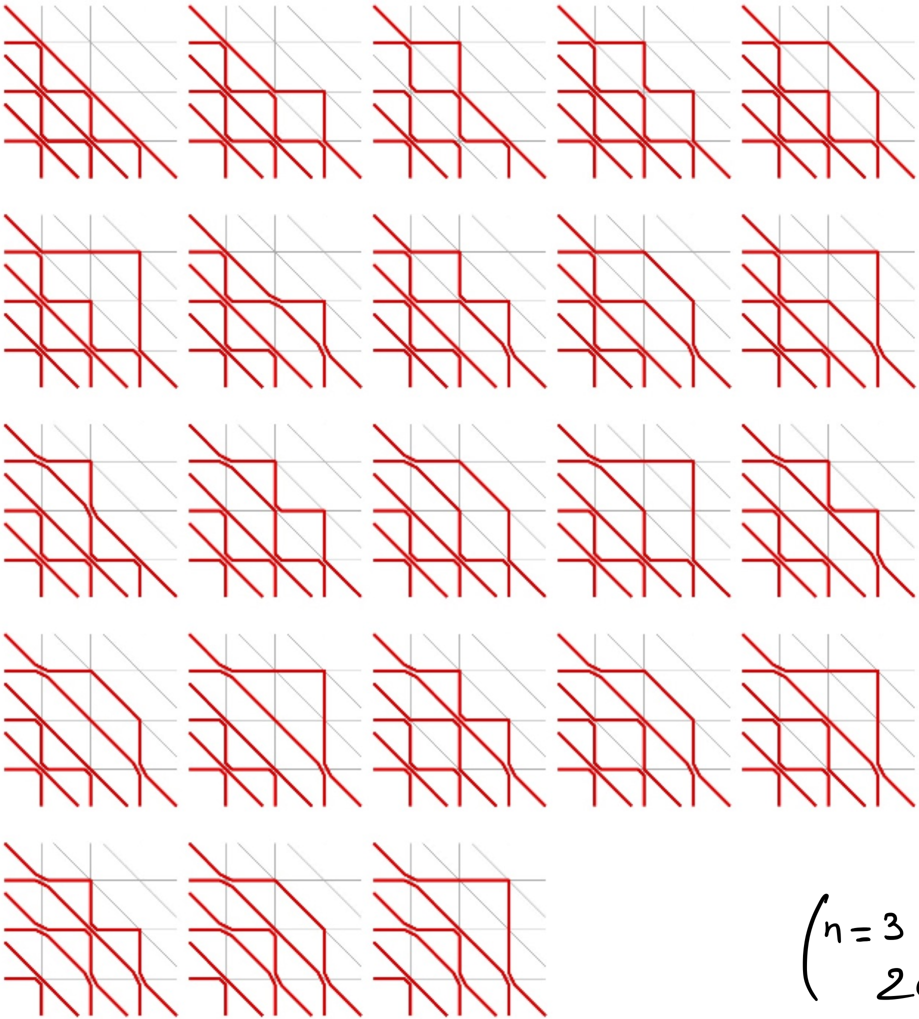
$$T_4(\mathcal{J}_n) = 1,\ 3,\ 23,\ 433,\ 19705,\ 2151843,\ \dots$$

Ex: $n=3$ $\quad \det\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 4 & 8 & 12 \end{pmatrix}\right] = 23$ Domino Tiling configurations $\rightarrow$





n = 3
holey square
Domino Tilings



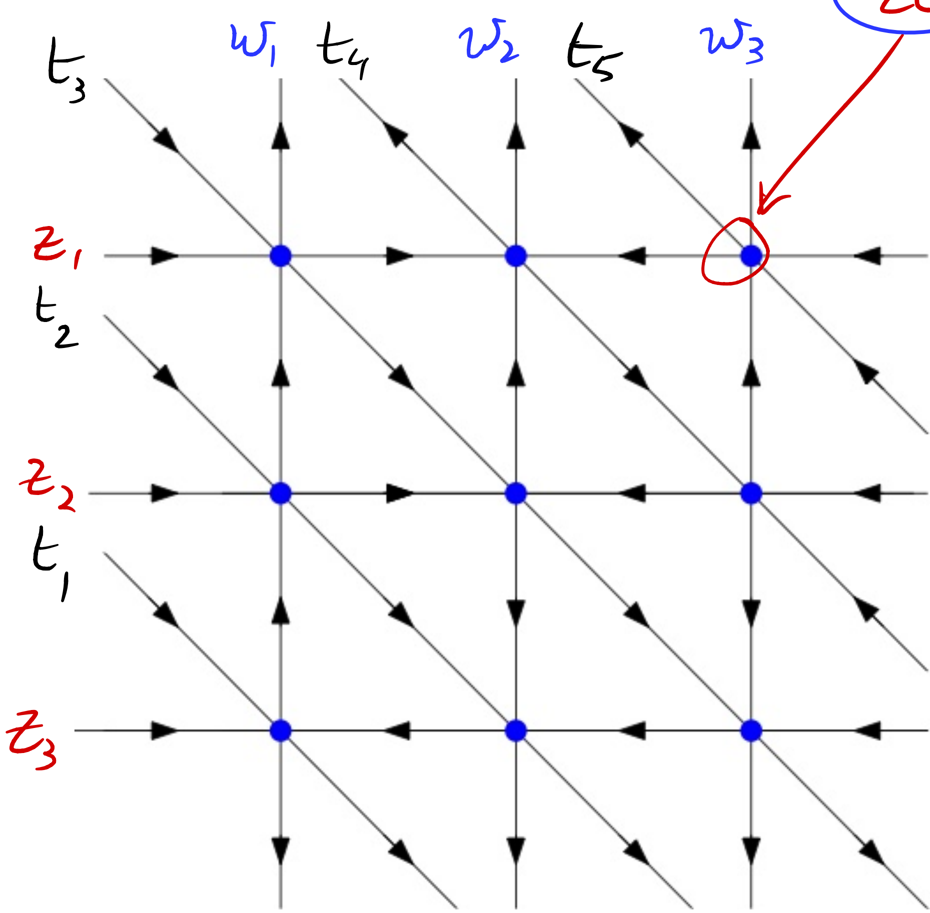
(n = 3 DWBC1
20V configurations)

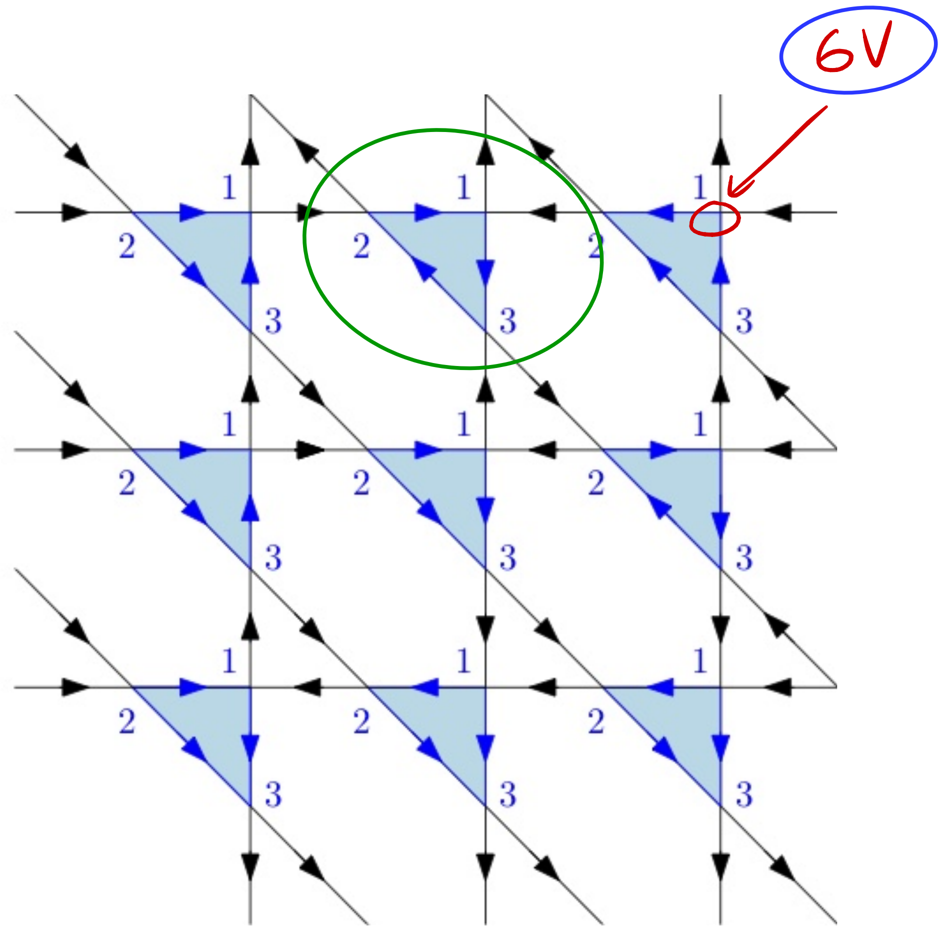
4. PROOF OF THE CORRESPONDENCE WITH 20V – DWBC1,2

idea

- use integrable weights for the 20V
- deform the line arrangement into a 6V
- use 6V results (Izergin-Korepin det)
- refinement

ICE MODEL ON THE KAGOME LATTICE





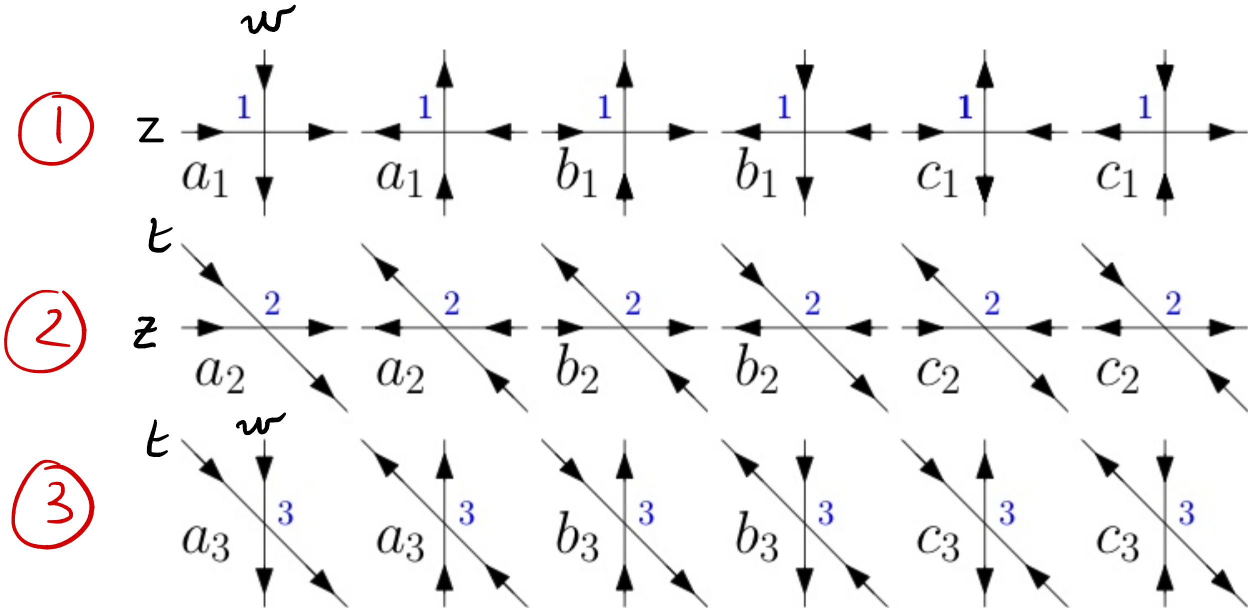
Triangular lattice ice

Kagome Lattice ice

20V weights ⟵ 6V weights on 3 sublattices 1,2,3

A lattice of KAGOME (Daikokuya, Kitashirakawa)

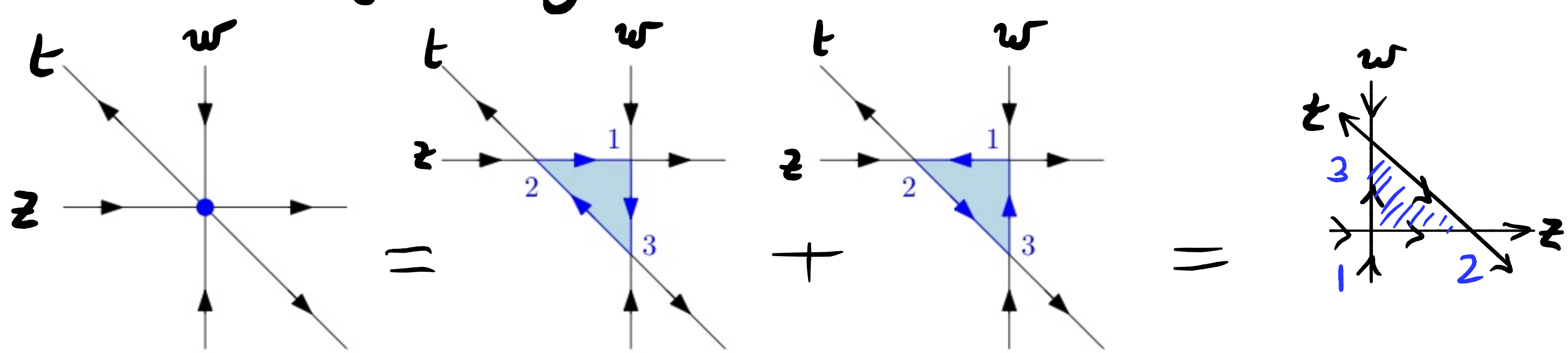
BOLTZMANN WEIGHTS



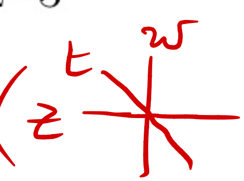
weights of the 6V models on the 3 sublattices

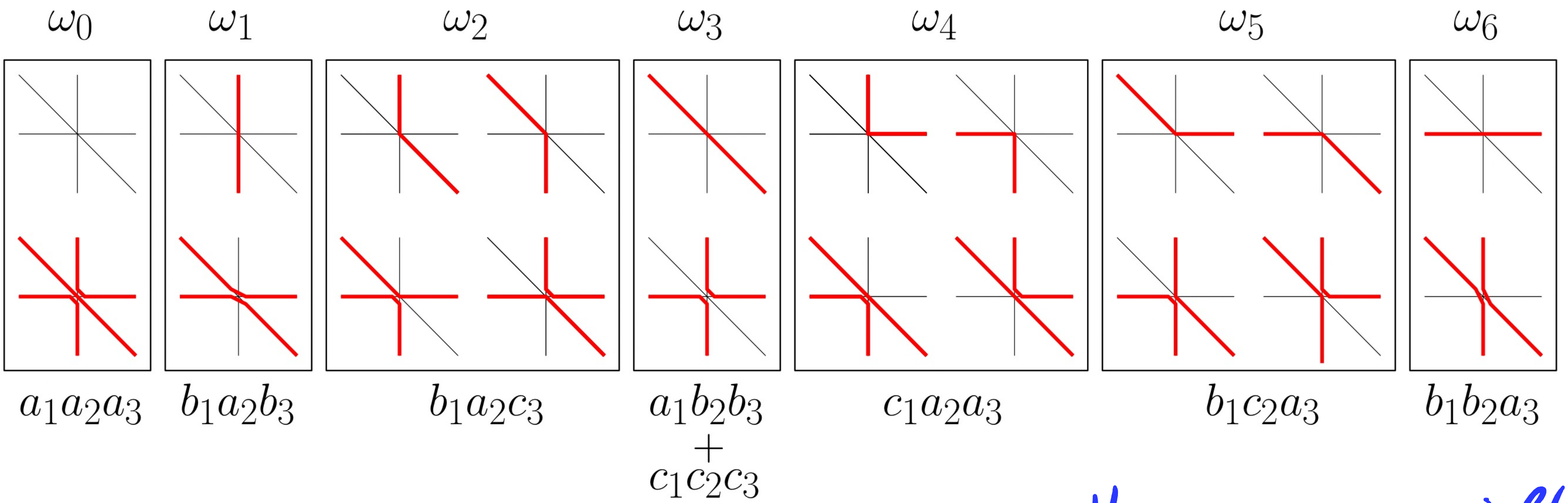
- 20V weights are given by sums over inner triangle configs

Example:



$$W = a_1 b_2 c_3 + c_1 c_2 b_3 = b_1 a_2 c_3$$

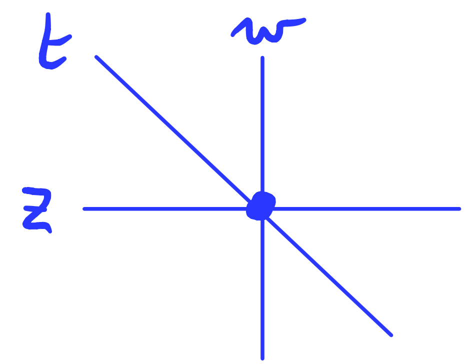
- Homogeneous case: 3 parameter family: $($  $, q)$



$$\omega_0 = \sin(\lambda+\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$
$$\omega_1 = \sin(\lambda-\eta)\sin\left(\frac{\lambda-\eta+\mu}{2}\right)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$
$$\omega_2 = \sin(2\eta)\sin(\lambda-\eta)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$
$$\omega_3 = \sin(2\eta)^3 + \sin(\lambda+\eta)\sin\left(\frac{\lambda-\eta+\mu}{2}\right)\sin\left(\frac{\lambda-\eta-\mu}{2}\right)$$
$$\omega_4 = \sin(2\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$
$$\omega_5 = \sin(2\eta)\sin(\lambda-\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)$$
$$\omega_6 = \sin(\lambda-\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)\sin\left(\frac{\lambda-\eta-\mu}{2}\right),$$

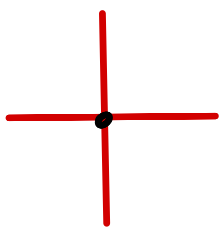
Homogeneous weights:

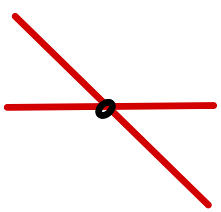
$$\begin{cases} q = e^{i\eta} \\ z = e^{i(\eta+\lambda)} \\ w = e^{-i(\eta+\lambda)} \\ t = e^{i\mu} \end{cases}$$

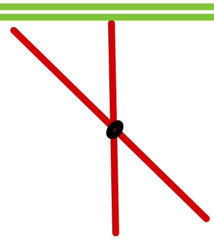


Remark: uniform weights $\omega_i = 1 \ \forall i$ are obtained for: $\boxed{\eta = \frac{\pi}{8} \quad \lambda = \frac{5\pi}{8} \quad \mu = 0}$

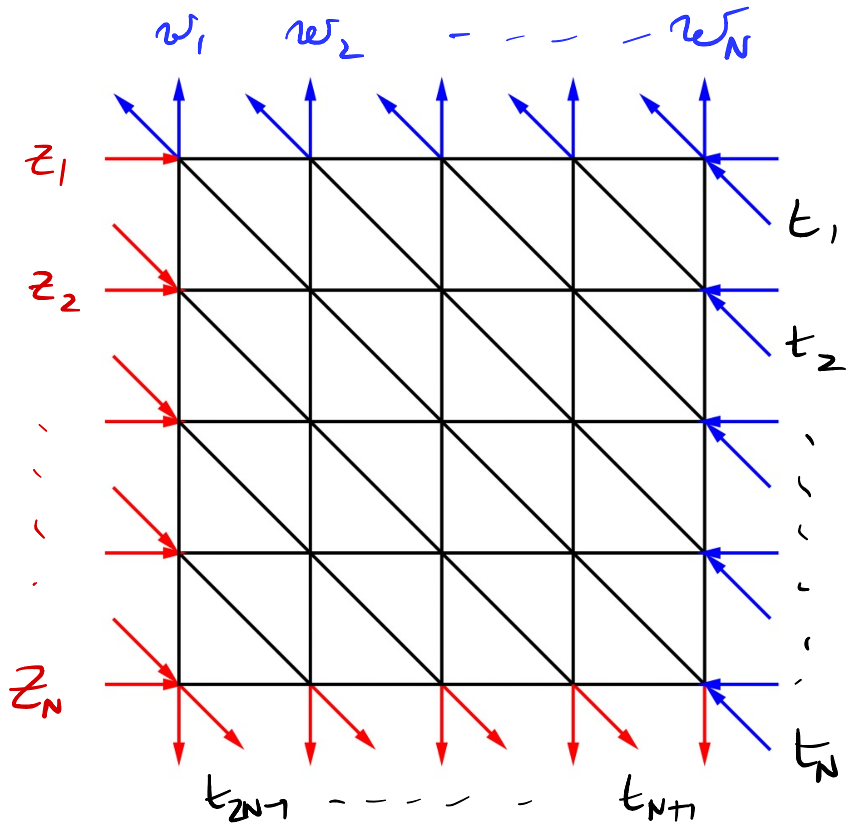
This corresponds to non-uniform weights on the 3 sublattice 6V models (up to an overall factor).

1  $a_1 = 1 \quad b_1 = \sqrt{2} \quad c_1 = 1$

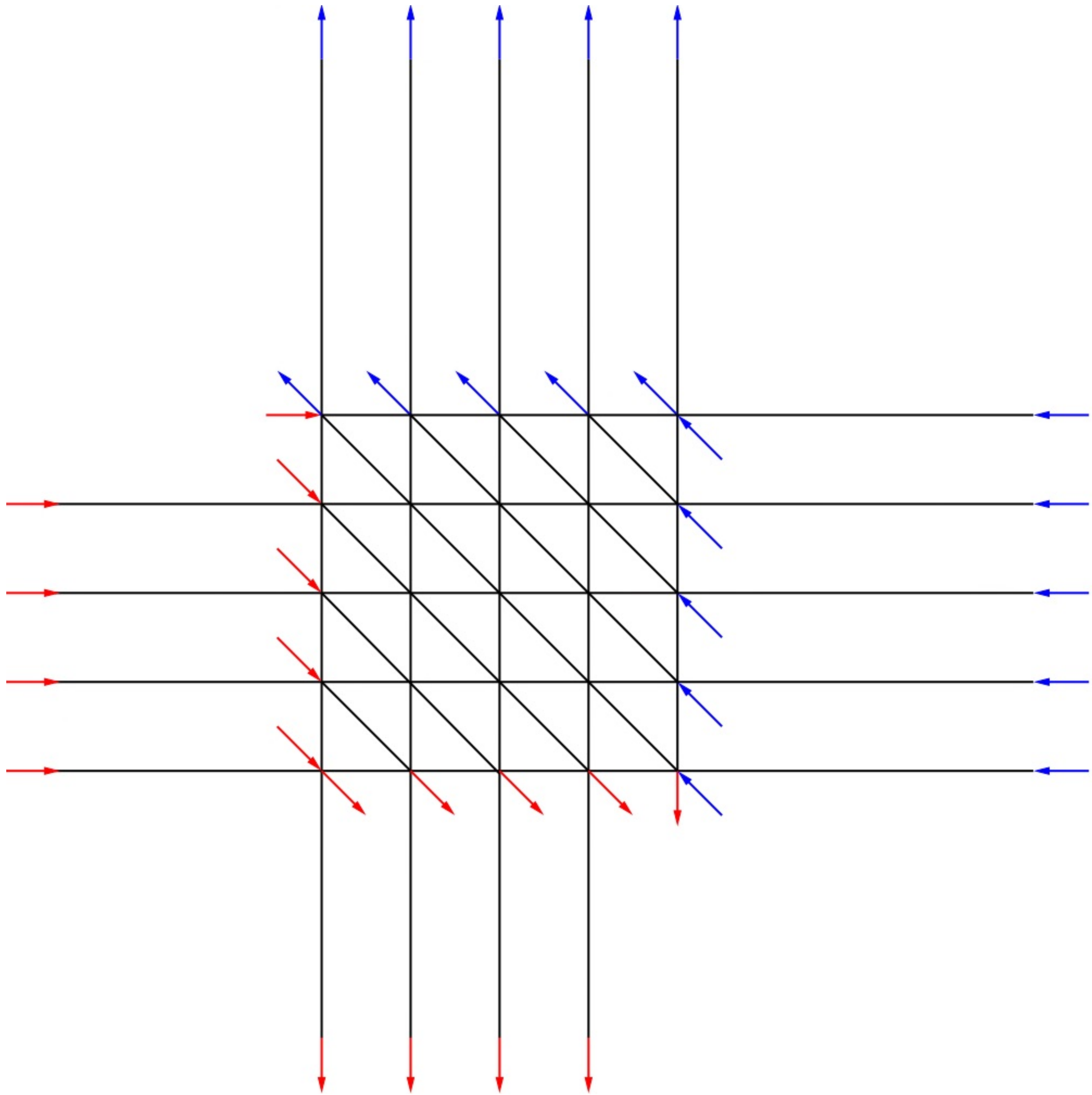
2  $a_2 = \sqrt{2} \quad b_2 = 1 \quad c_2 = 1$

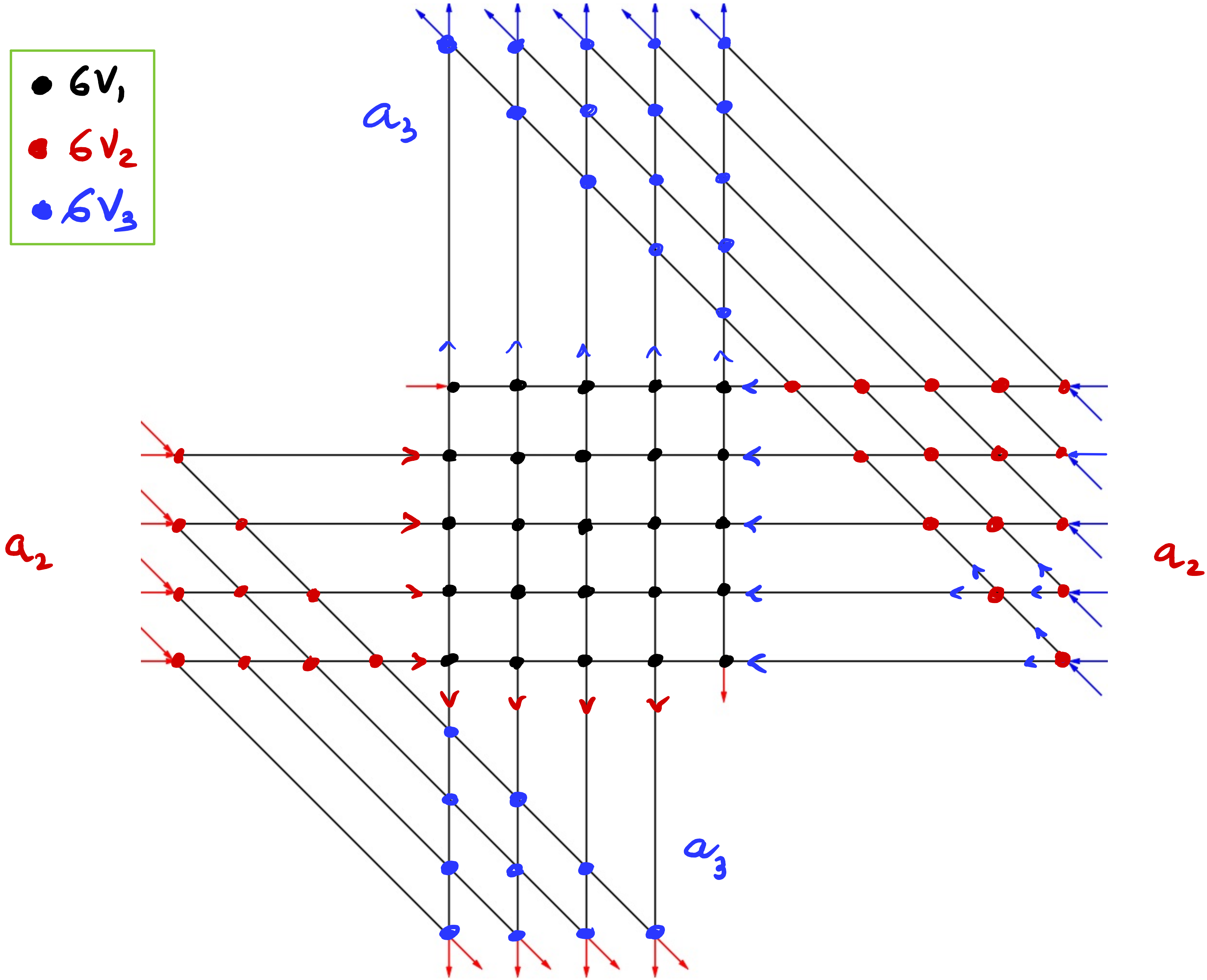
3  $a_3 = \sqrt{2} \quad b_3 = 1 \quad c_3 = 1$

TRANSFORMATION INTO a 6V MODEL

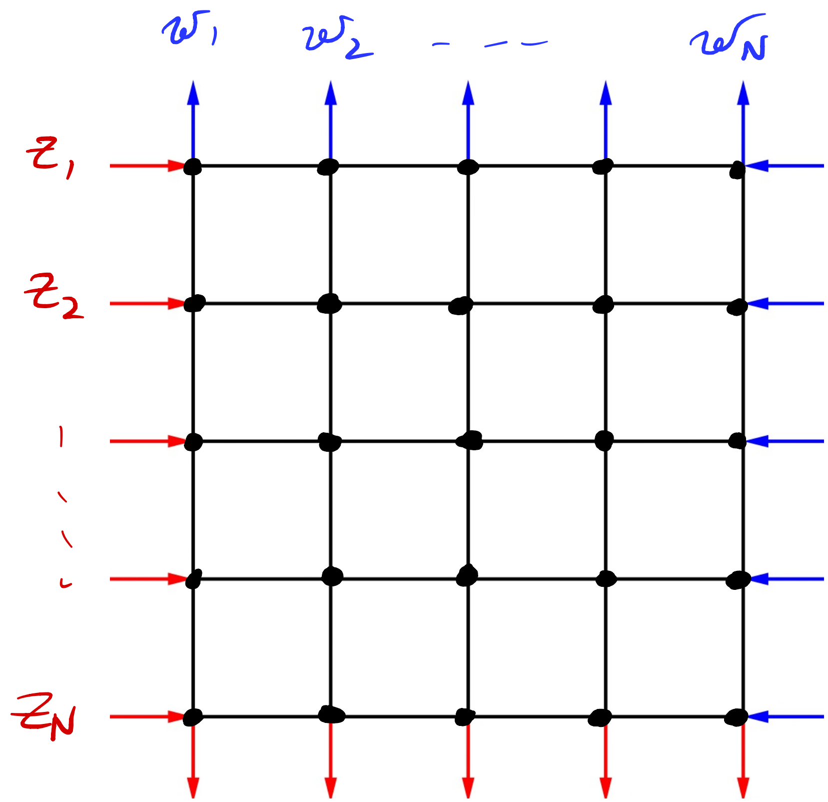


20V DWBC-2 (integrable weights).





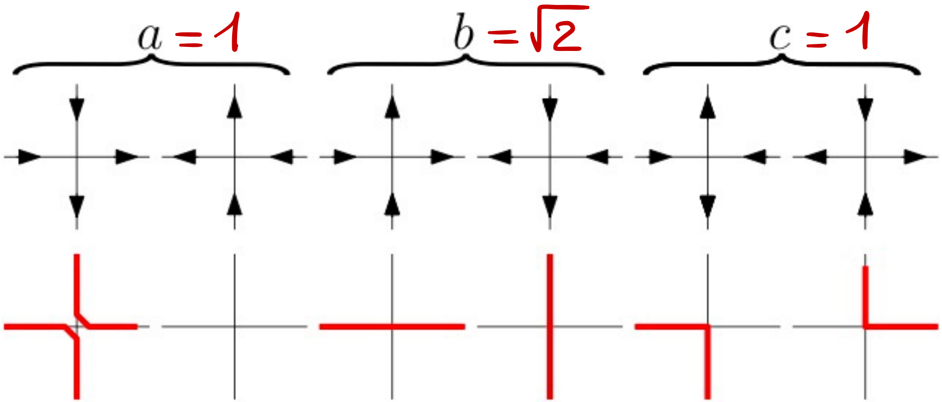
$6V_1$
$6V_2$
$6V_3$
a_3
a_2
a_2
a_3



6V DWBC (sublattice 1 only).

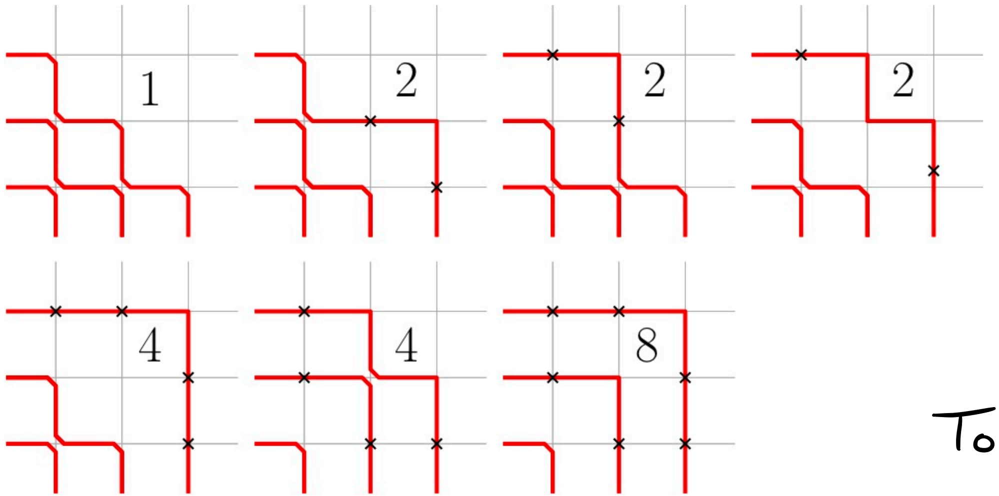

20種の野菜と6種の果実
野菜汁60%+果汁40%=100%
20種の野菜と6種の果実
野菜汁60%+果汁40%=100%
KAGOME
野菜生活
100
ベリーサラダ
1食分の野菜
KAGOME
野菜生活
100
ベリーサラダ
1食分の野菜
20
6

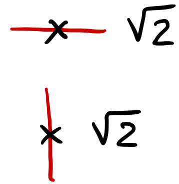
Thm [PDF, E. Guitter 19] The partition function of the 20V model with all weights $= 1$ is equal to that of the 6V model with weights $(a, b, c) = (1, \sqrt{2}, 1)$ and DWBC



Example of size $n=3$

20V-DWBC1 vs 6V aka ASM





(b weights)

Total $=$ 23 **APM$_1$** of size 3

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

7 ASM of size 3.

Thm [PDF, E. Guitter 19] The partition function of the 20V model with all weights = 1 is equal to that of the 6V model with weights $(a,b,c)=(1,\sqrt{2},1)$ and DWBC

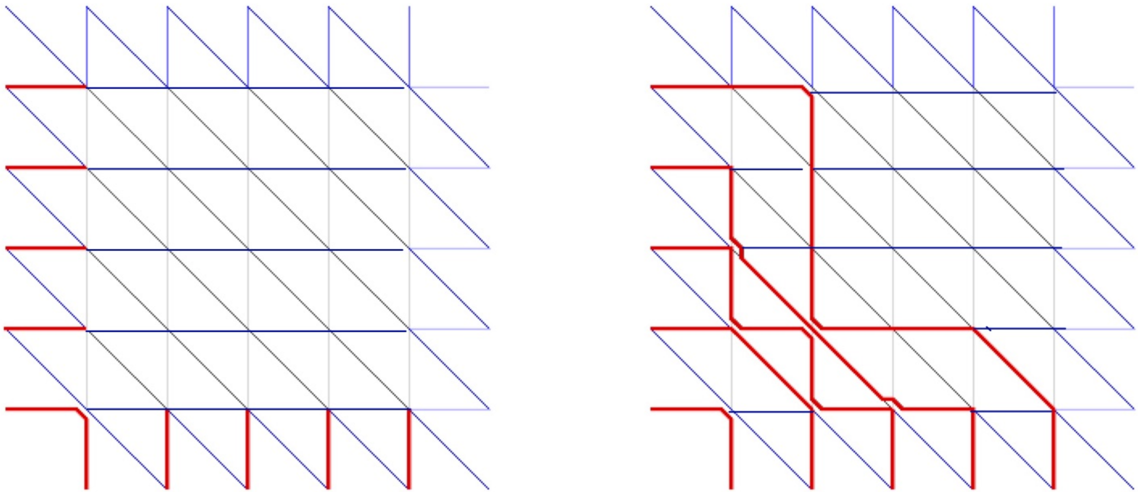
Then use classical result by Korepin-Izergin for the 6V-DWBC and spectral parameters $(z_1 \ldots z_n, w_1 \ldots w_n)$

$$Z_{6VDWBC}(z_1 \ldots z_n, w_1 \ldots w_n) = \frac{\prod_{i=1}^{n} c(z_i, w_i) \prod_{i,j=1}^{n} a(z_i,w_j)\, b(z_i,w_j)}{\prod_{1\leq i<j\leq n} (z_i - z_j)(w_i - w_j)} \det_{1\leq i,j\leq n}\left\{\frac{1}{a(z_i,w_j)\, b(z_i,w_j)}\right\}$$

→ Limiting procedure → same det as holey square DT!
(cf Behrend, PDF, Zinn Justin)

→ Refinements

5. The DWBC 3 Conjectures



DWBC 3 $\qquad b_n = 1, 3, 29, 901, \ldots$

The DWBC 3 Conjectures

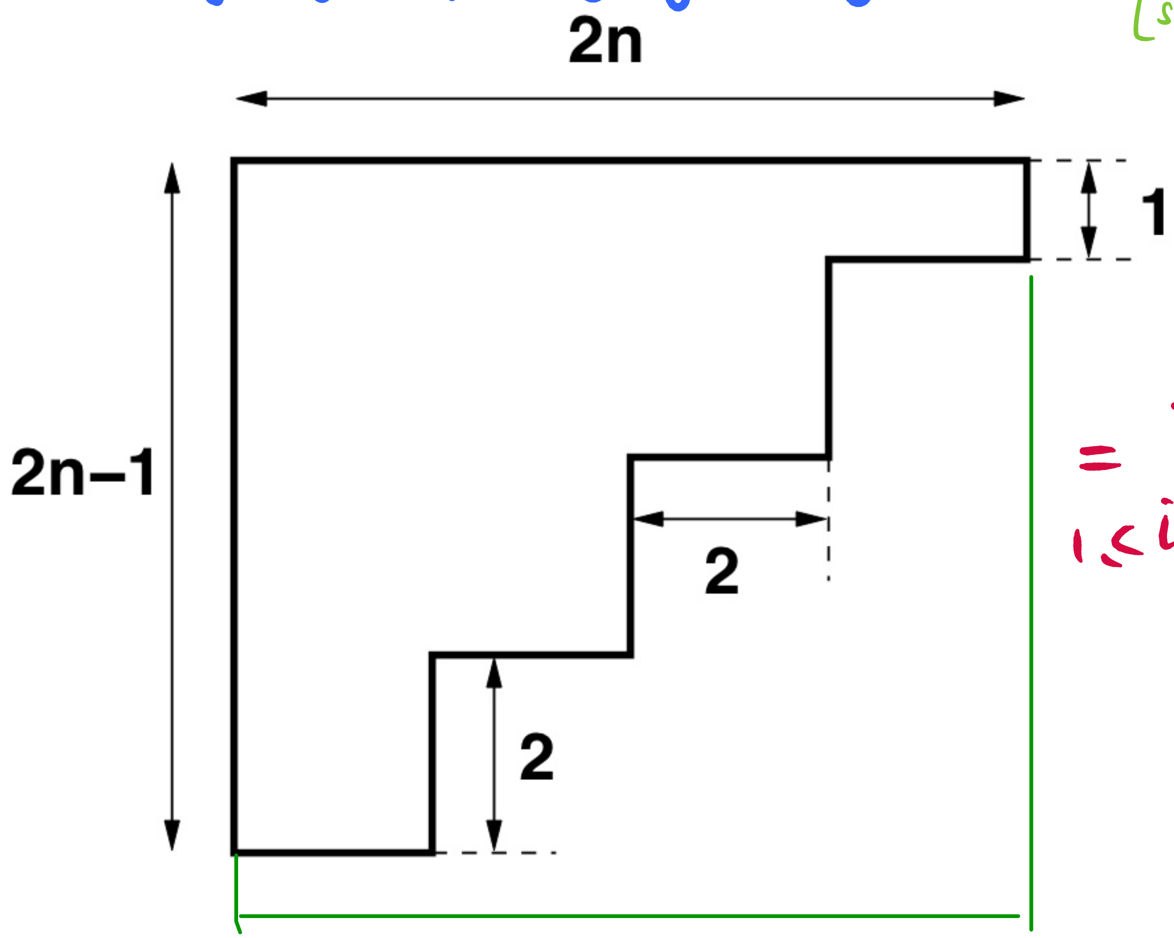
[OEIS for B_n] $\longrightarrow$ Domino Tilings of a $2n \times 2n$ square

$$= 2^n b_n^2$$

$b_n = 1, 3, 29, 901 \ldots$ [Temperley-Fisher '61]

[Pachter 97] proof of integrality of b_n: found a Domino Tiling interpretation [see also Ciucu]

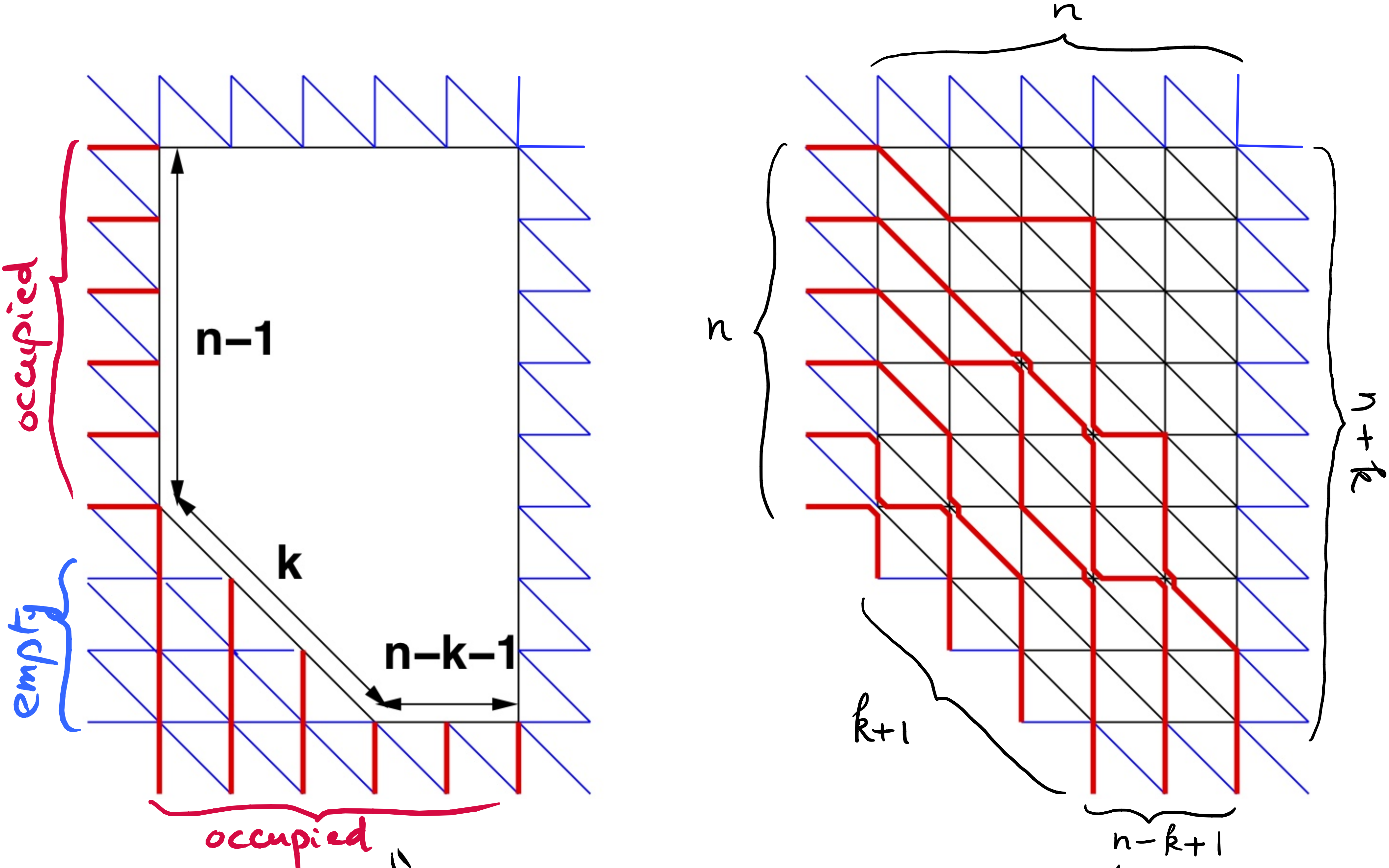
$b_n =$



$$= \prod_{1 \le i < j \le n} \left(4\cos^2 \frac{\pi i}{2n+1} + 4\cos^2 \frac{\pi j}{2n+1} \right)$$

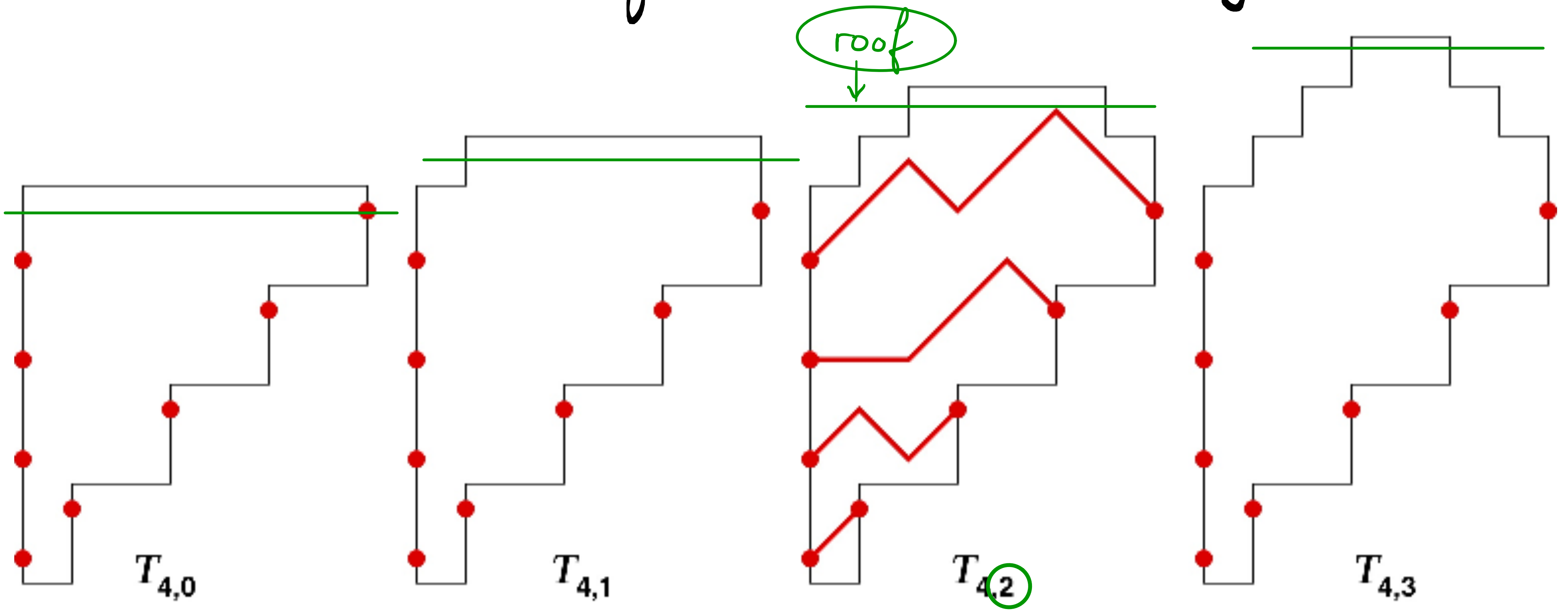
Conjecture 1. [PDF– E. Quitter 19] The configurations of the 20V–DWBC3 model on an $n \times n$ grid are counted by the Domino Tilings of Pachter's triangle

But we can do better....

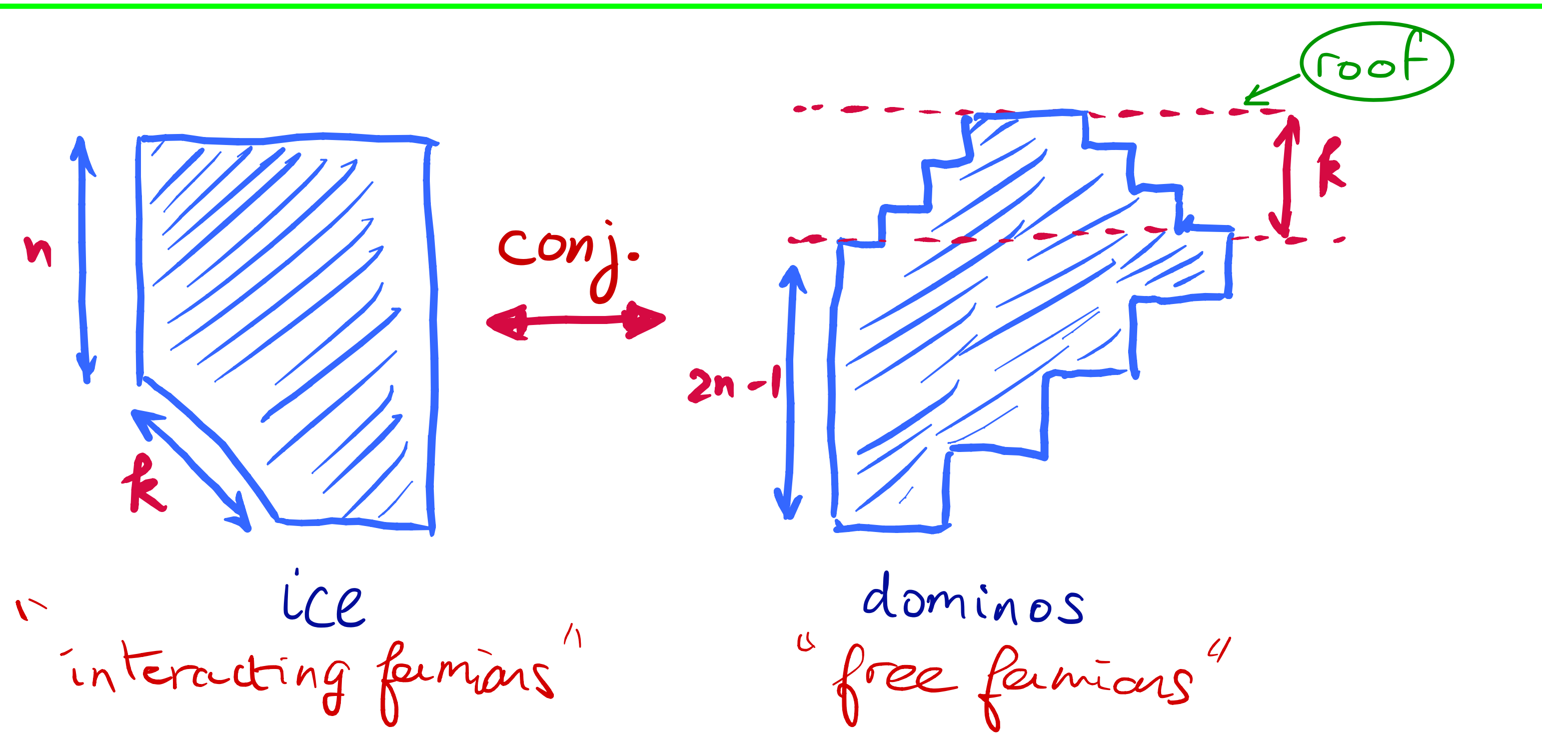


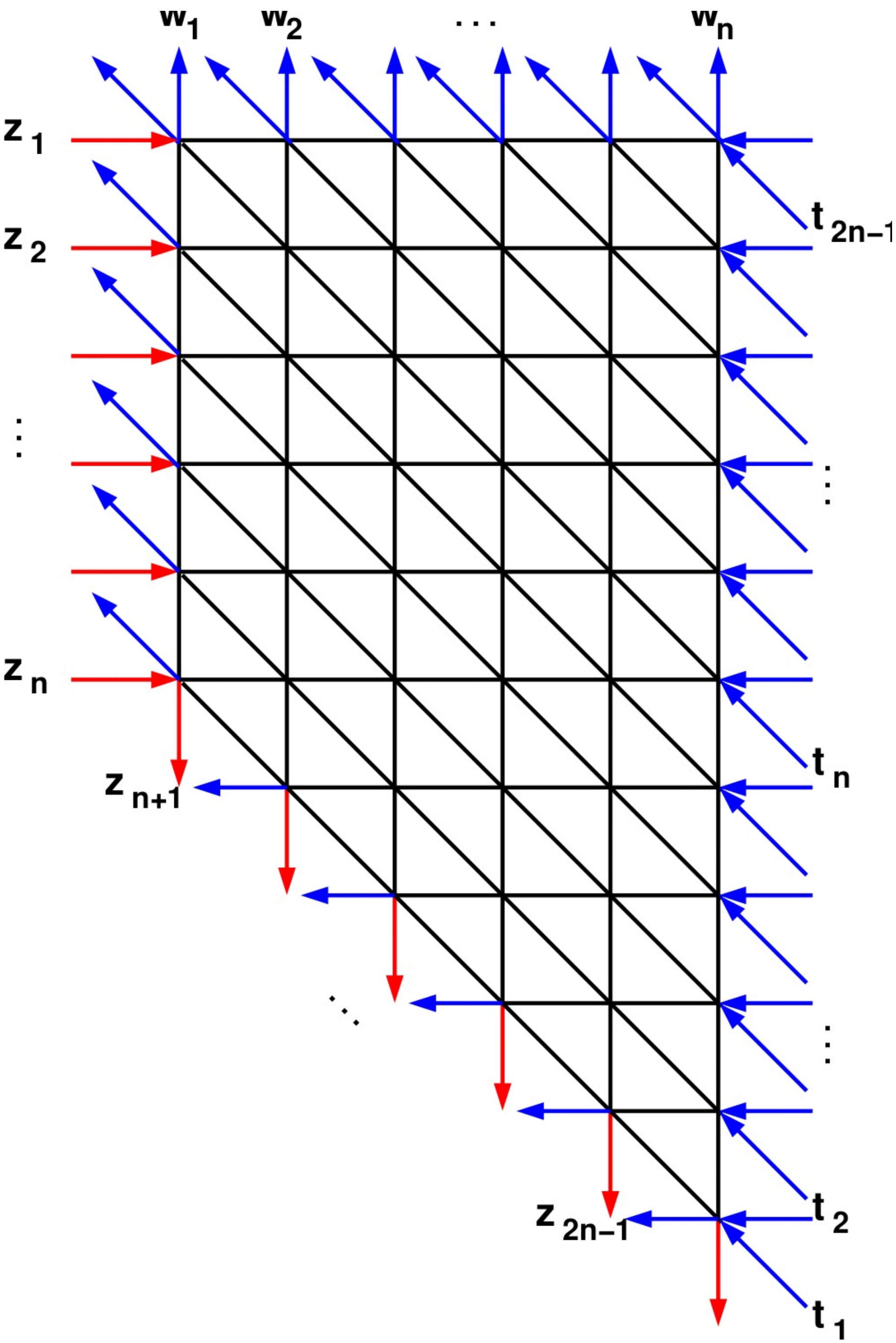
“Pentagon of triangular ice”

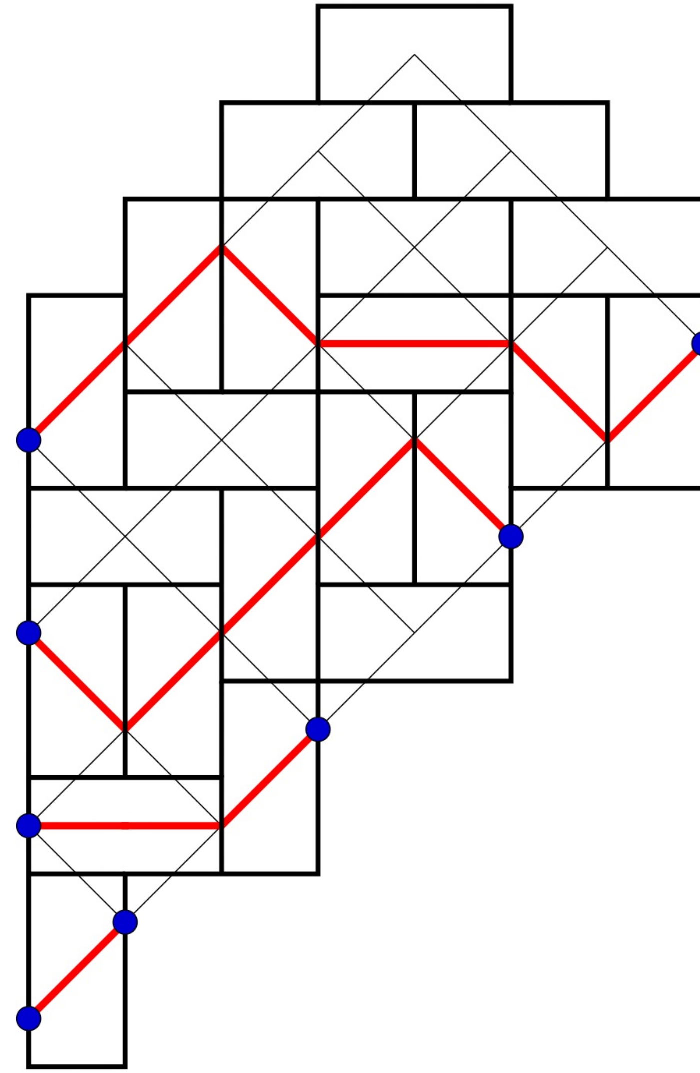
"raise the roof" above Catalan's triangle



Conjecture 2. [PDF+E. Guitter 19] The number of configurations of triangular ice in a pentagon w/ DWBC3 is equal to that of domino tilings of Pachter's raised triangle







DT of AztecTriangle$_n$

20V-DWBC 3 on Q_n

$$\boxed{Z_n = 1, 4, 60, 3328, \ldots}$$

Thm [Di Francesco 2021]

$$\boxed{Z^{20V\text{-}DWBC3}_{Q_n} = Z^{DT}_{AT_n}} = \det_{0\le i,j\le n-1}\left(\frac{1+u}{(1-uv)^2-v(1+u)^2}\Bigg|_{u^i v^j}\right)$$

Conjecture 3

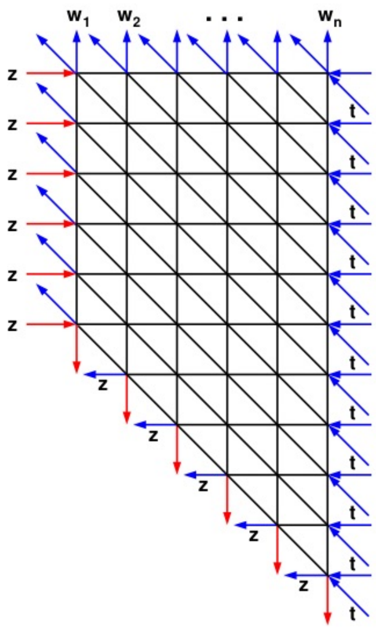
$$\boxed{Z^{20V\text{-}DWBC3}_{Q_n} = 2^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \frac{(4j+2)!}{(n+2j+1)!}}$$

$= 1, 4, 60, 3328, 678912, \ldots$

+ refinements

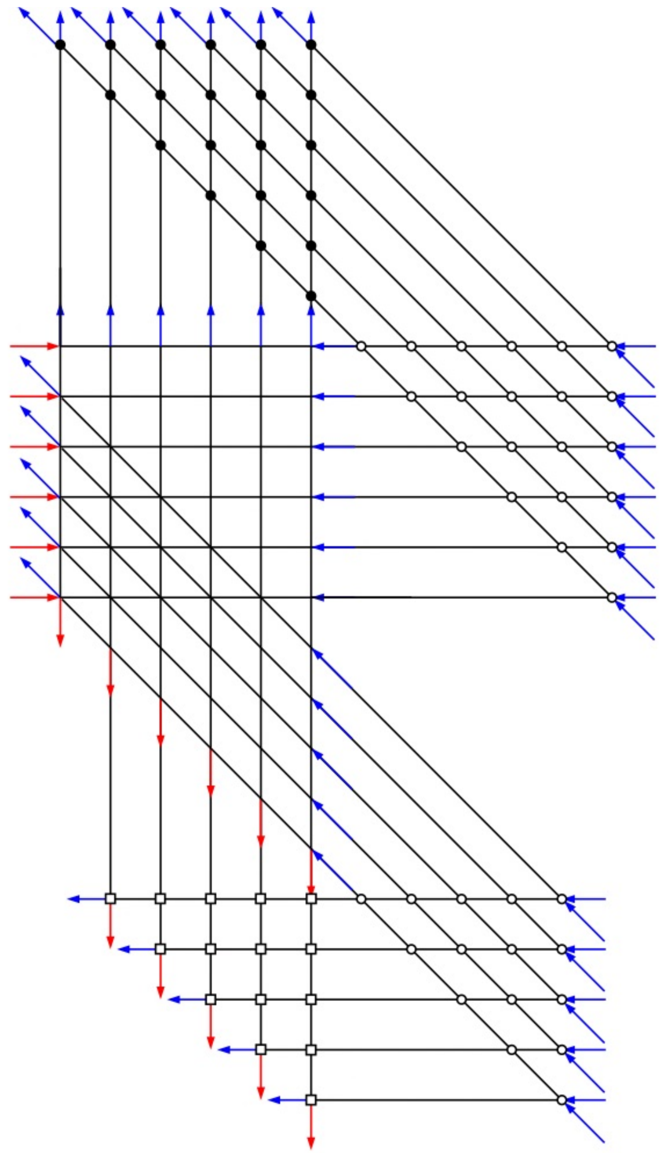
Progress [CIUCU '21] [Krattenthaler '21] [Trilai + PDF '21]

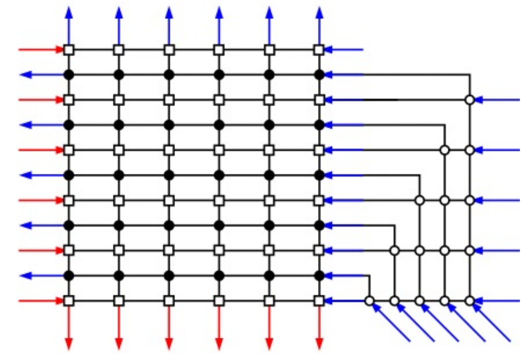
- Proof of Thm Along the same lines



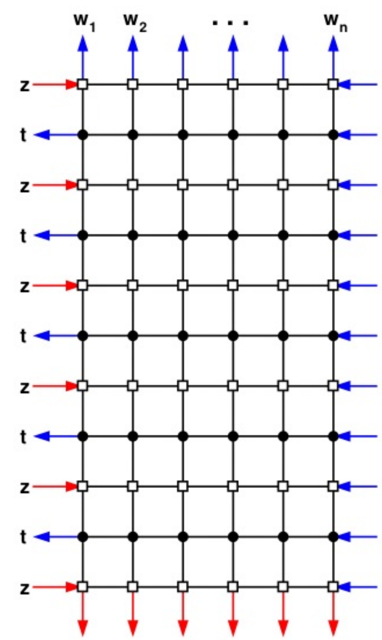
(a)

20V-DWBC3 on Q_n





(c)



6V with U-turn Boundary

[Kuperberg, Tsuchiya]

↓

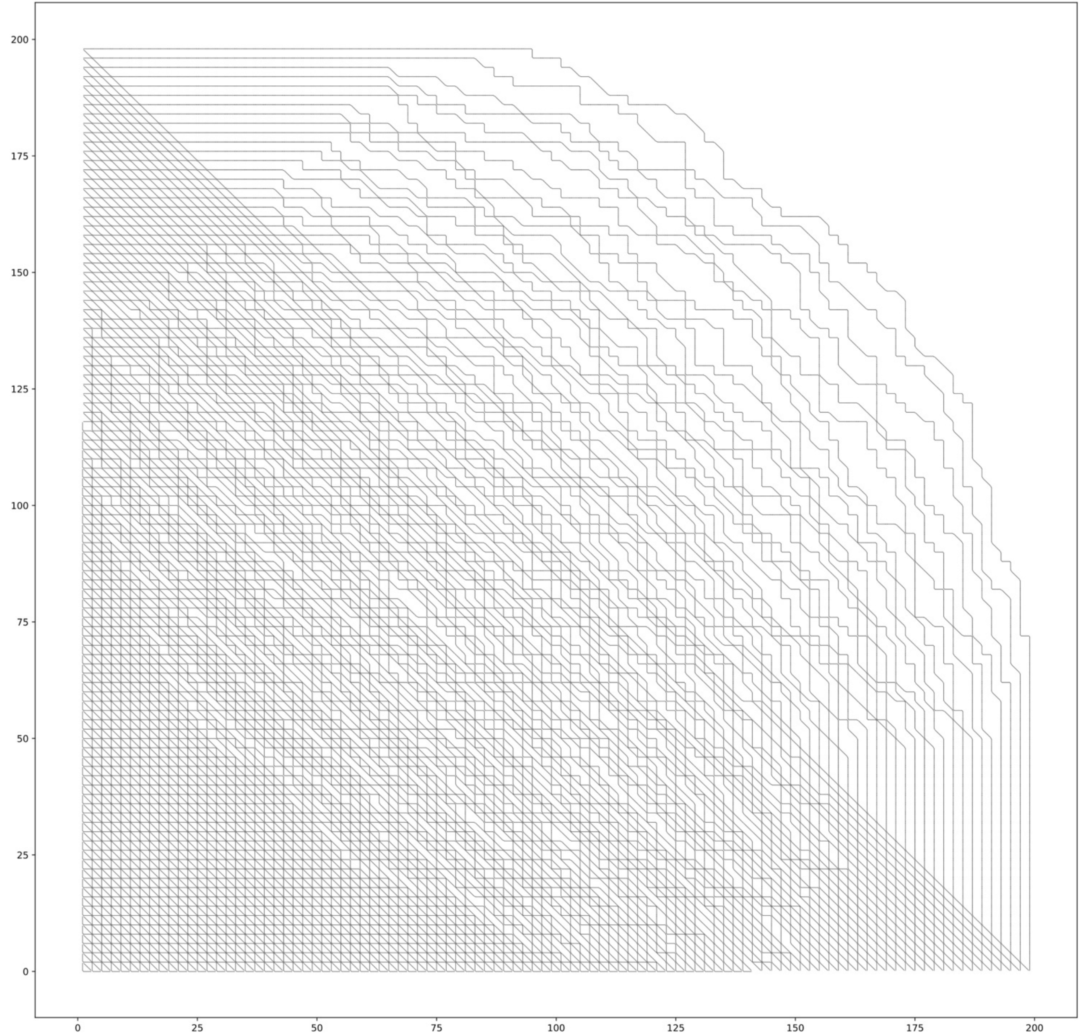
determinant!

6. LIMIT SHAPE : THE ARCTIC PHENOMENON

- large size N ; typical configuration exhibits "frozen" domains / "liquid" domains

"frozen" domains $\longrightarrow$ regularly ordered paths

"liquid" domains $\longrightarrow$ disordered paths



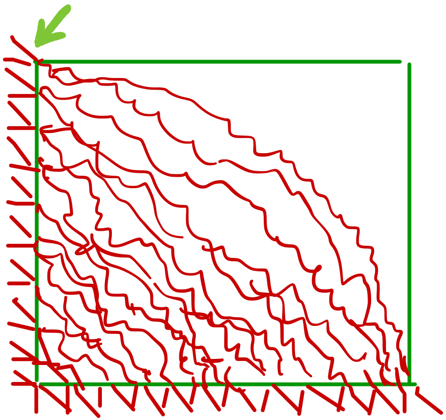
DWBC1

uniform weights

N= 200

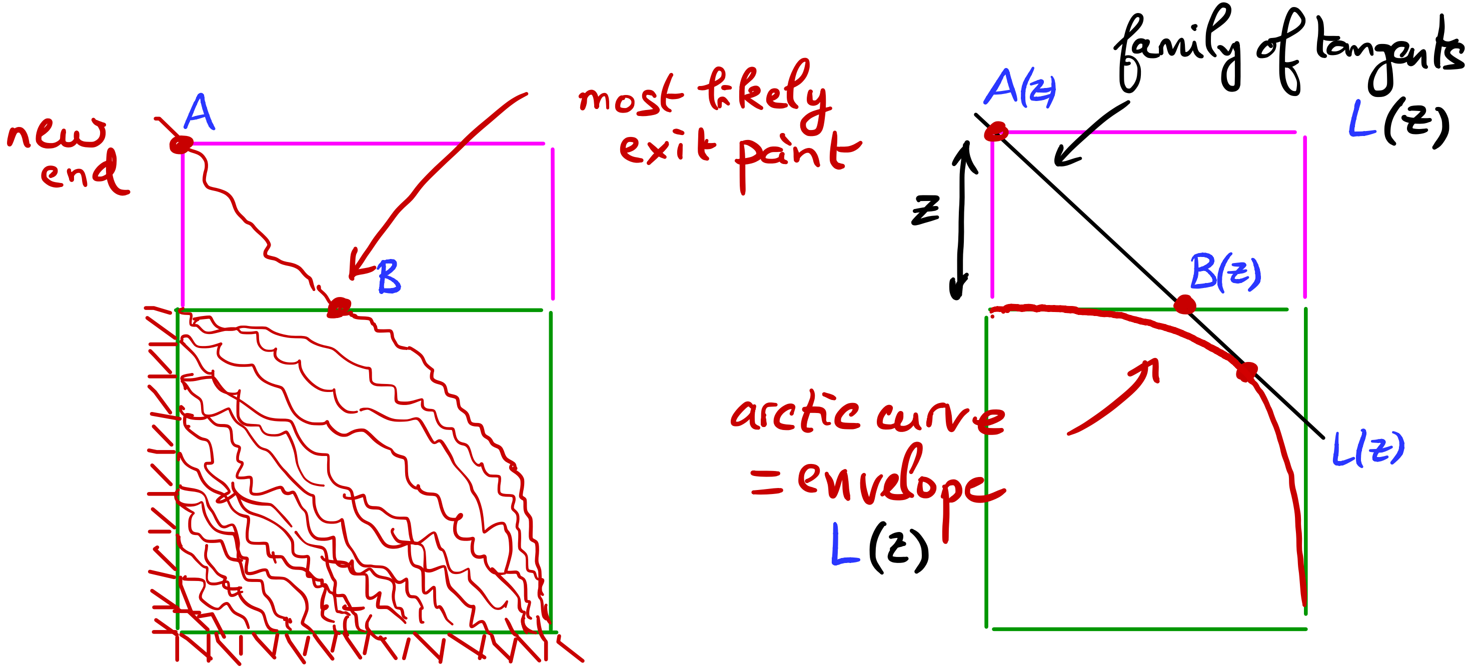
ARCTIC PHENOMENON (20V DWBC1)

- Typical shape of a large configuration
  → use "tangent method" [Colomo - Sportiello '16]

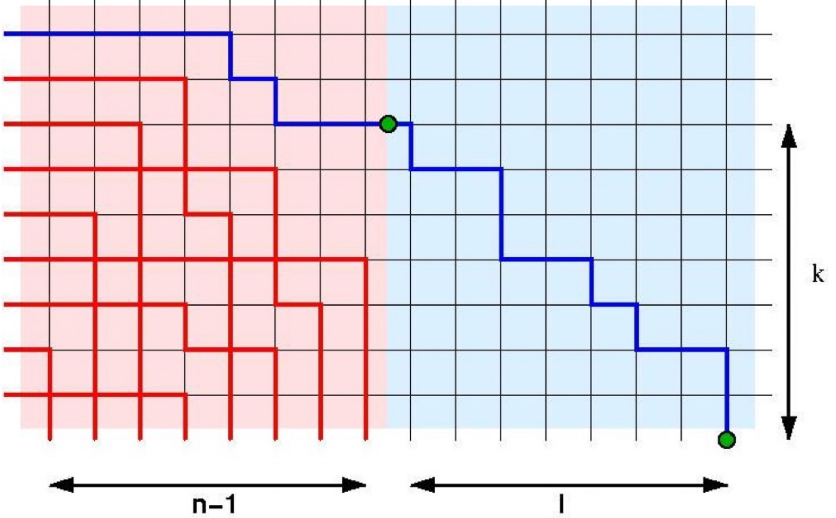


- modify last path exit point
- use this new path as probe for the limit shape

ARCTIC PHENOMENON (20V DWBC1)



6V DWBC

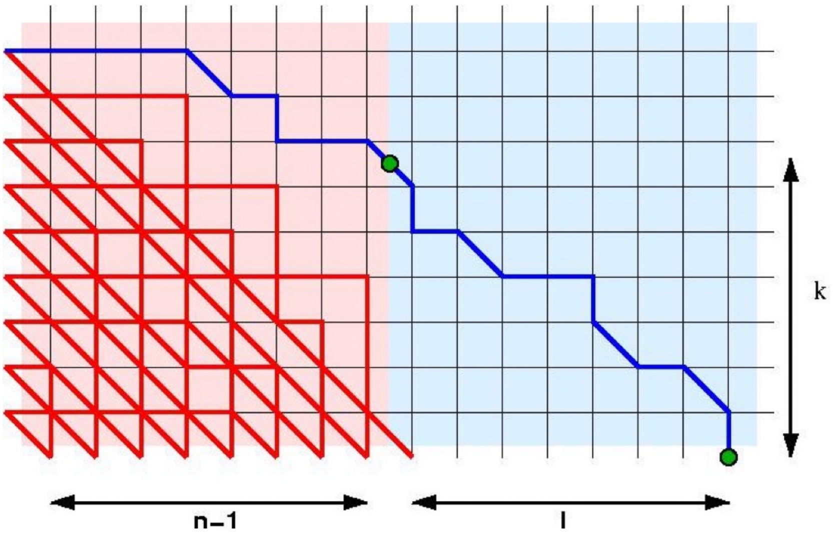


Recipe

compute both the pink and blue partition functions!

+ large n, ℓ, k estimates

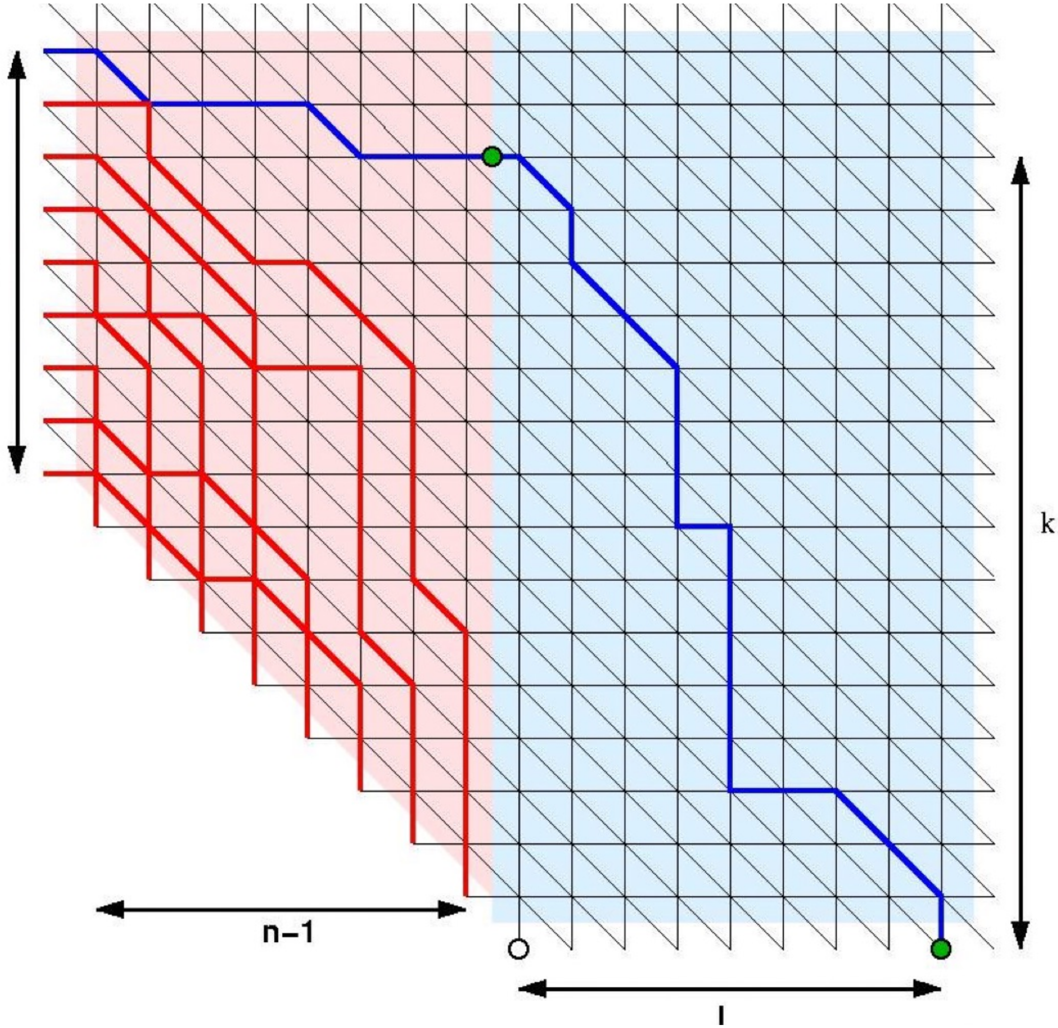
20V DWBC1



+ saddle-point solution

$\Rightarrow k(\ell)$

+ envelope of lines thru $(\ell, 0)$ & $(0, k)$

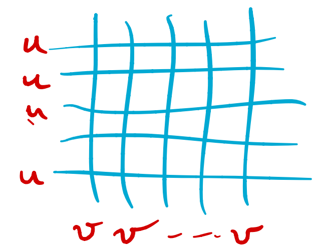


20V DWBC 3
(quadrangle)

• Partition Function:

from 1k determinant formulas
($z_i \to u$, $w_j \to v$)

$$\frac{Z_{N+1}\, Z_{N-1}}{Z_N^2} + \frac{1}{N^2}\frac{\partial^2}{\partial u\,\partial v}\log Z_N = 0$$

$Z_N =$ 

$$Z_N = e^{-N^2 f} \Rightarrow \boxed{\frac{\partial^2 f}{\partial u\,\partial v} = e^{-2f}}$$

(2D Liouville/Toda eq)

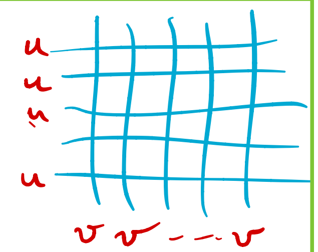
• Partition Function:

from IK determinant formulas

$(z_i \to u \,,\ w_j \to v)$

$$\frac{Z_{N+1}\, Z_{N-1}}{Z_N^2} + \frac{1}{N^2} \frac{\partial^2}{\partial u \partial v} \log Z_N = 0$$

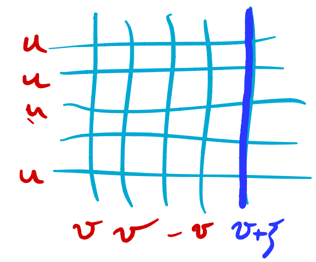
$Z =$



$$Z_N \simeq e^{-N^2 f} \Rightarrow \boxed{\frac{\partial^2 f}{\partial u \partial v} = e^{-2f}}$$

(2D Liouville/Toda eq)

$Z(\xi) =$



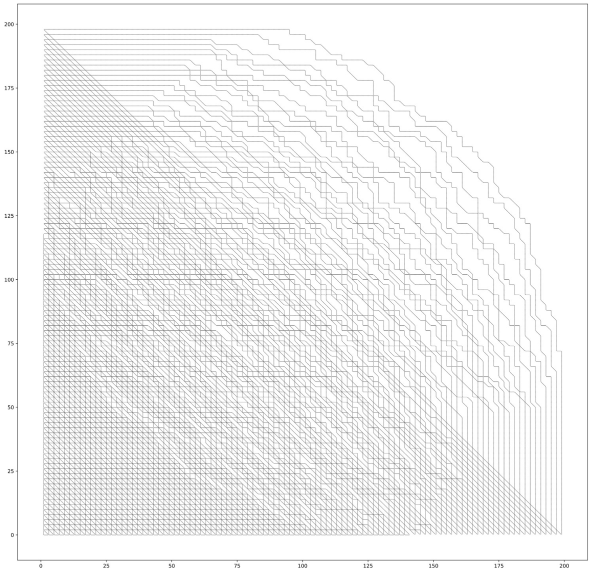
• One-point Function:

$(z_i \to u \,,\ w_i \to v_{1 \leq i \leq N-1} \,,\ w_N \to v+\xi)$

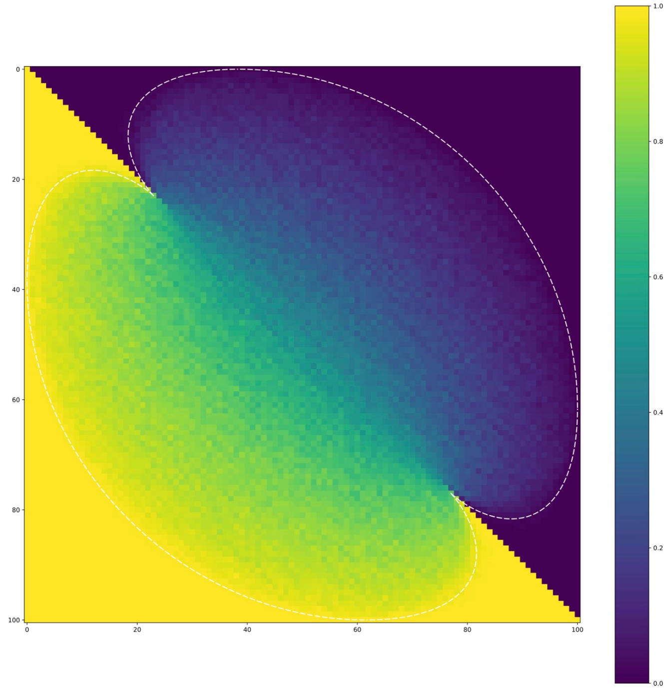
$$H_N = (N-1)! \frac{Z_N(\xi)}{Z_N(0)} \Rightarrow \frac{Z_{N+1}\, Z_{N-1}}{Z_N^2} \quad \frac{H_{N+1}}{H_N} + \frac{1}{N} \partial_u \log H_N = 0$$

$$H_N \simeq e^{-N\varphi} \Rightarrow \boxed{\partial_u \varphi = e^{-2f-\varphi}}$$

20V-DWBC1 uniform weights



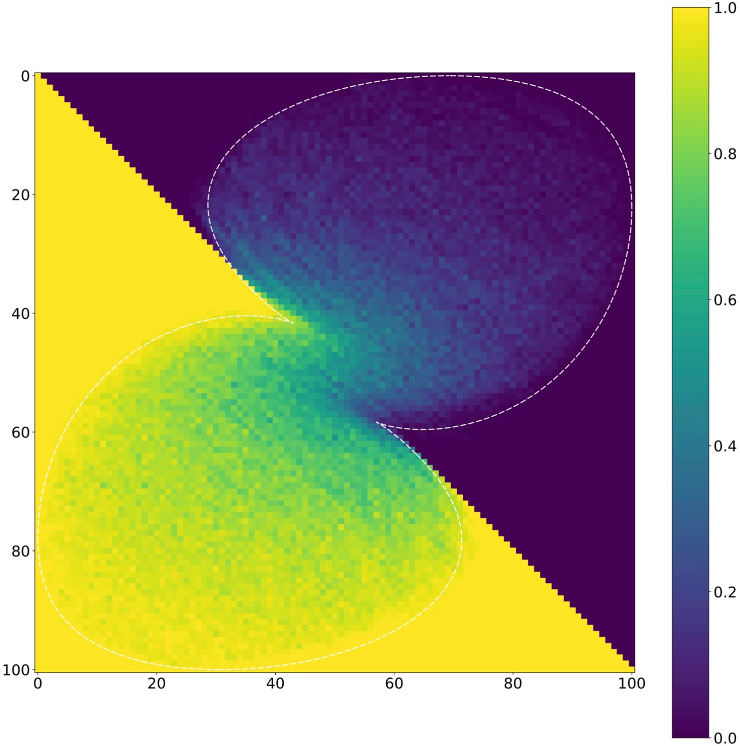
N=200



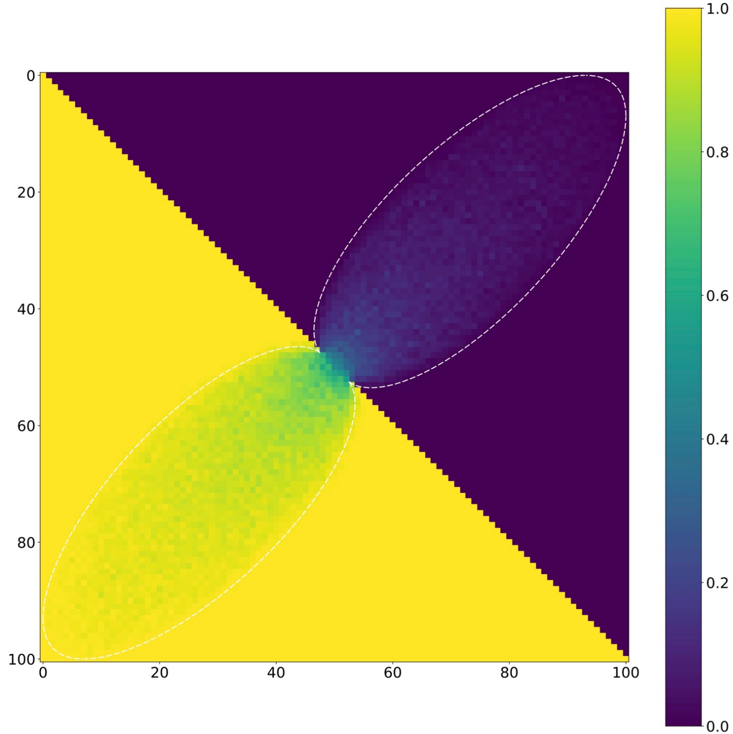
N=100

20V-DWBC1 – Non-uniform integrable weights

$(\omega_0, \omega_1, \ldots, \omega_6)$

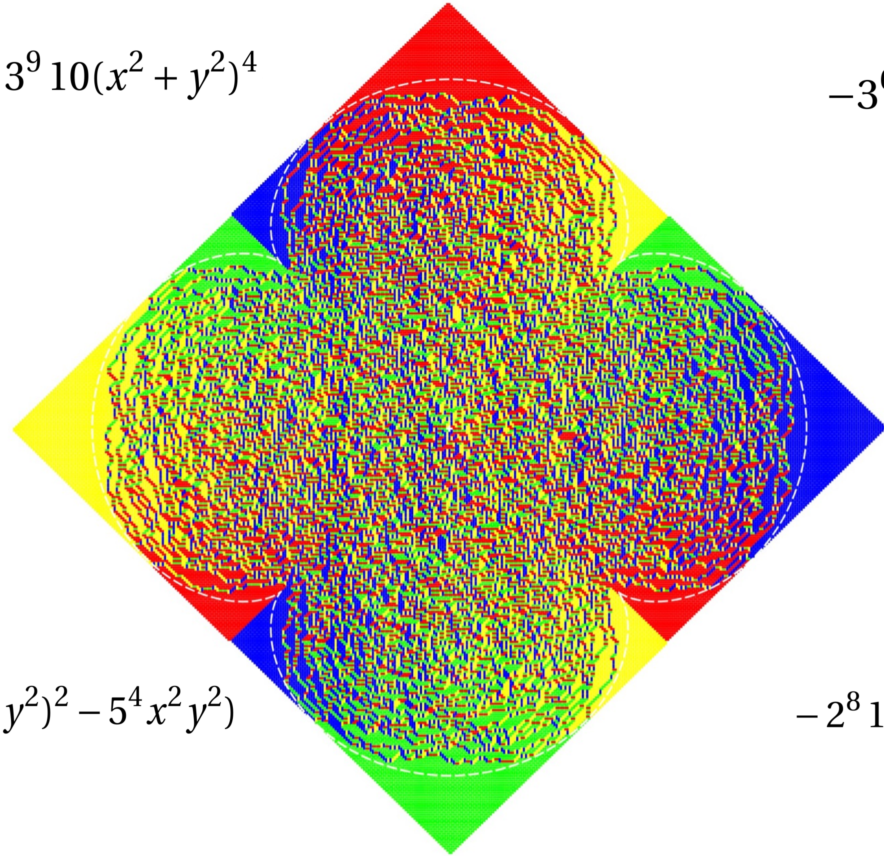


N=100



N=100

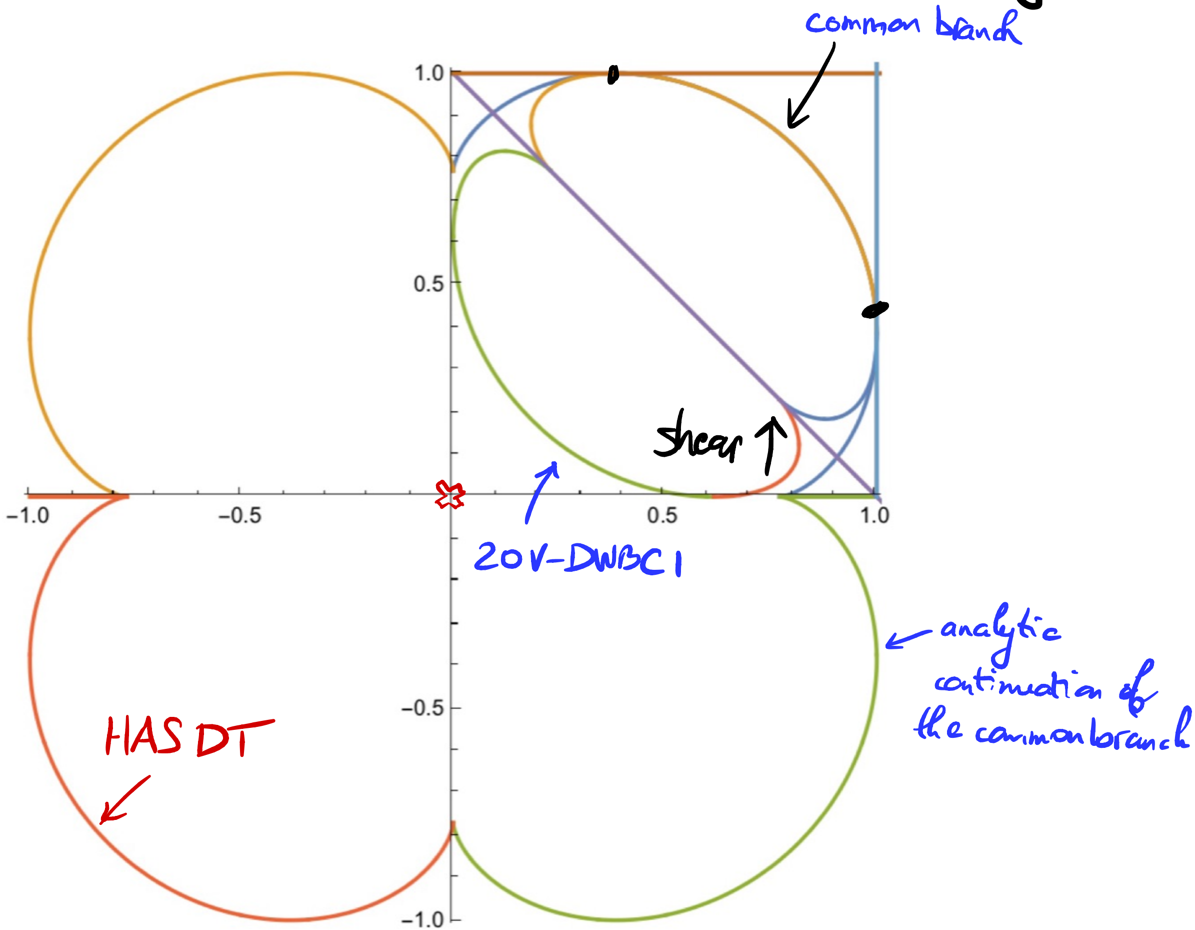
Holey Aztec square domino tilings (uniform weights)

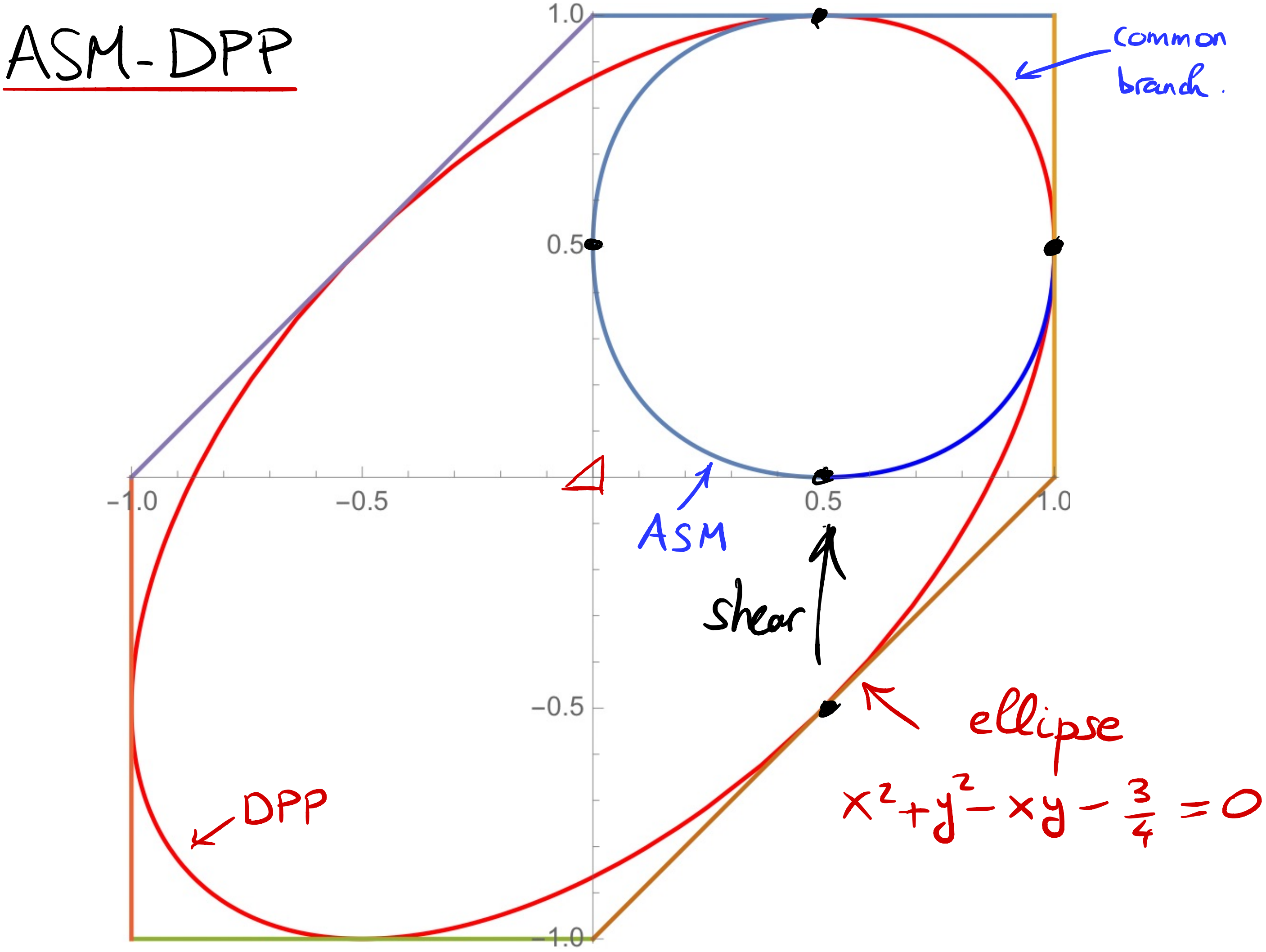


$$3^{11}(x^2+y^2)^5 + 3^9\,10(x^2+y^2)^4 - 3^6\,5(x^2+y^2)^3$$

$$+6^2\,20(73(x^2+y^2)^2 - 5^4x^2y^2) - 2^8\,15(x^2+y^2) - 2^{12} = 0\,.$$

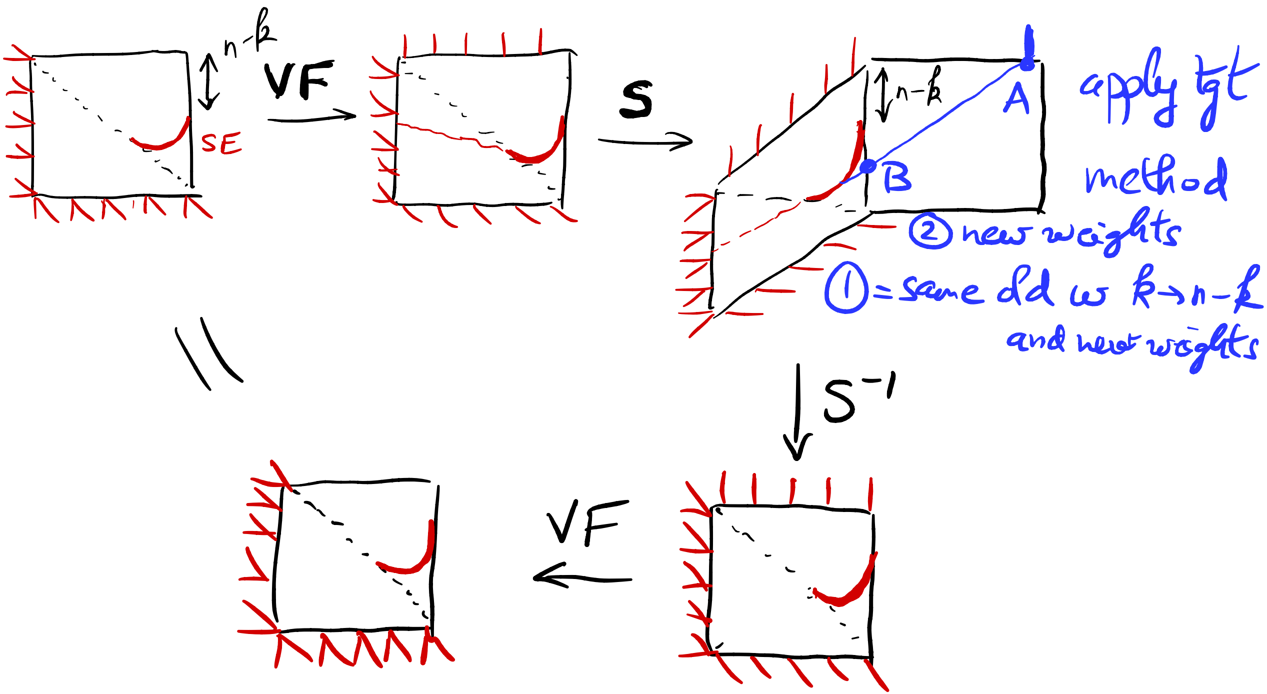
APM – holey Aztec Domino Tiling

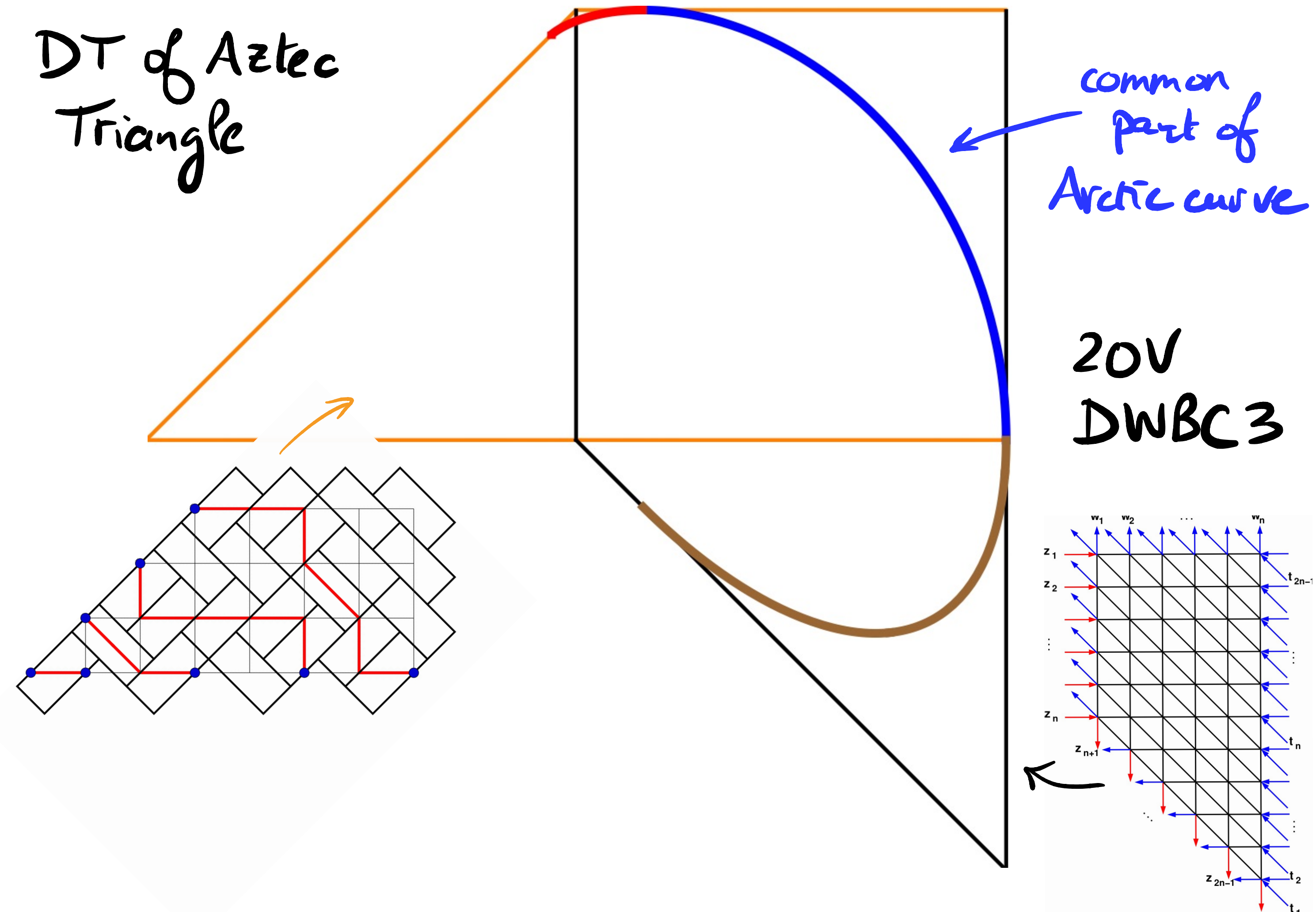




ASM-DPP
common branch.
ASM
shear
ellipse
$x^2+y^2-xy-\frac{3}{4}=0$
DPP
1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0

THE SHEAR TRICK





DT of Aztec Triangle
common part of Arctic curve
20V DWBC3
w_1 w_2 ... w_n
z_1
z_2
z_n
z_{n+1}
z_{2n-1}
t_{2n-1}
t_n
t_2
t_1

7. CONCLUSION

- Triangular ice does have interesting combinatorics!

  osculating lattice paths

- Arctic Phenomenon DWBC1, 2, 3 have one!

  → use tangent method

  → use refinements and connection to 6V

  U-6V

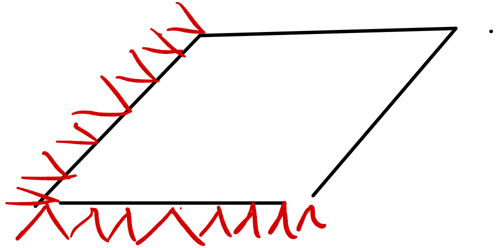
) Analytic predictions

- non-analyticity / shear phenomenon for "interacting fermions".

- related loop models? (webs?) (RS-like conjecture?)
- Classify the "good" boundary conditions?

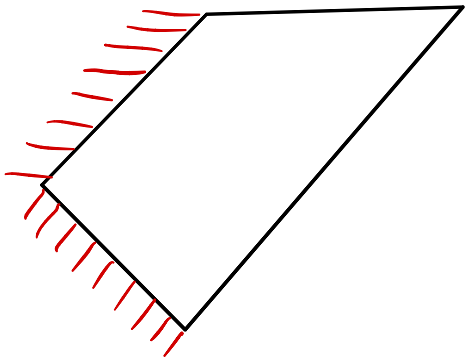
SOLVABLE CASES SO FAR

DWBC1,2
(Lozenge)



Z_n^{6V}

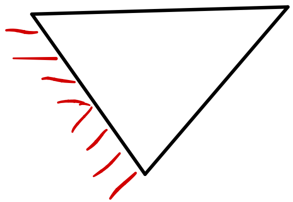
DWBC3
(quadrangle)



$Z_n^{6V\text{-}U}$

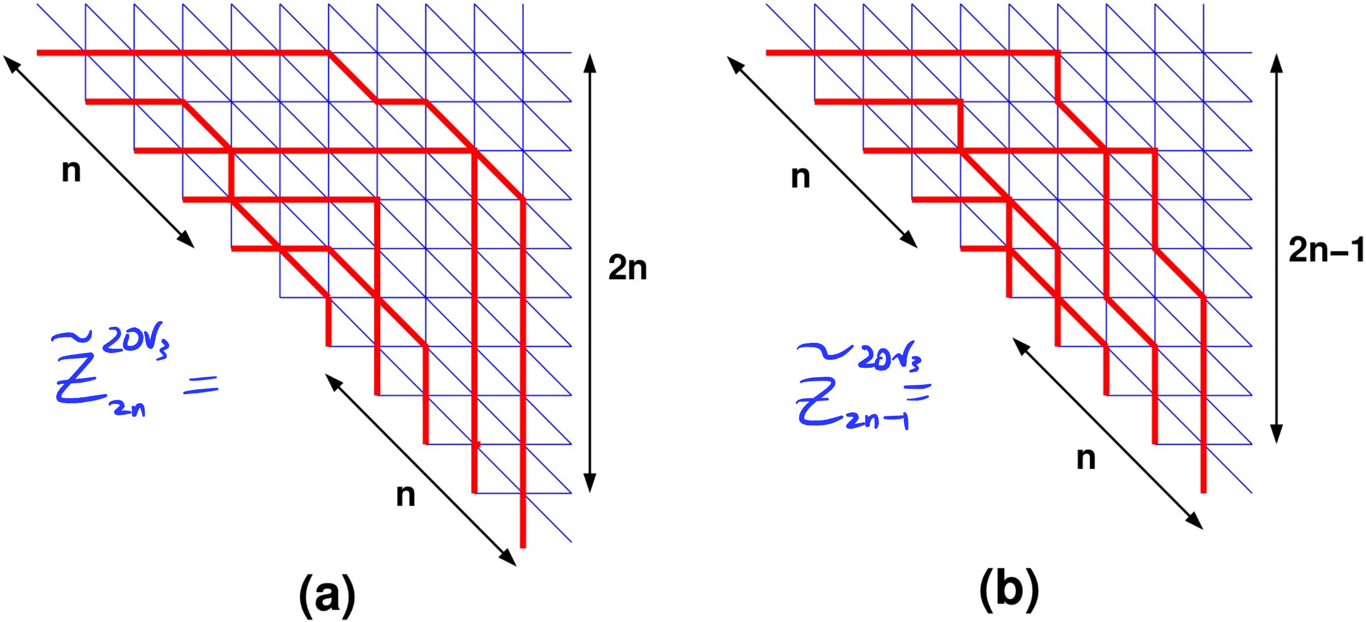
DWBC3
(triangle)

NEW



$2^{n(n\pm 1)/2}\, Z_n^{6V}$

Another 20V DWBC3 model



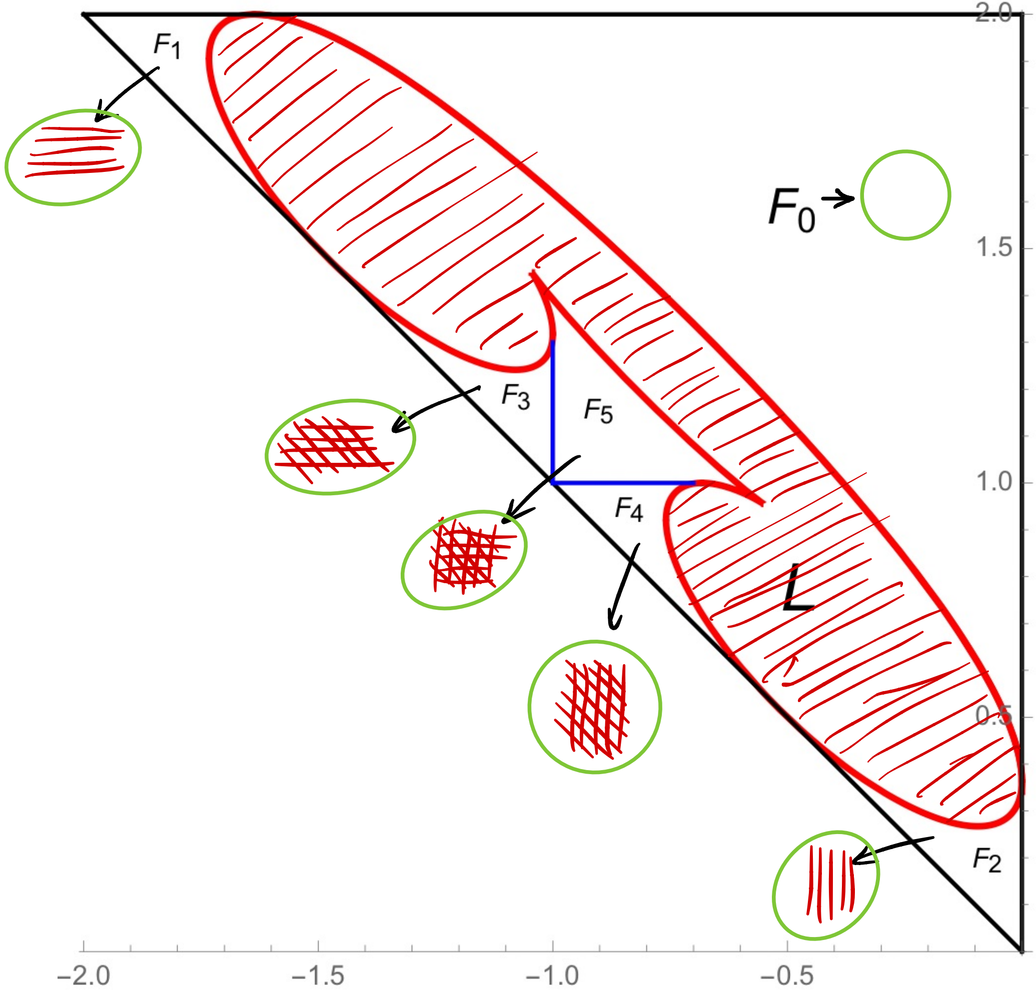
$\tilde{Z}^{20V_3}_{2n} =$

$\tilde{Z}^{20V_3}_{2n-1} =$

THM

$$\tilde{Z}^{20V_3}_{2n} = 2^{n(n+1)/2}\, Z_n^{20V\text{-}DWBC_1}$$

$$\tilde{Z}^{20V_3}_{2n-1} = 2^{n(n-1)/2}\, Z_n^{20V\text{-}DWBC_1}$$

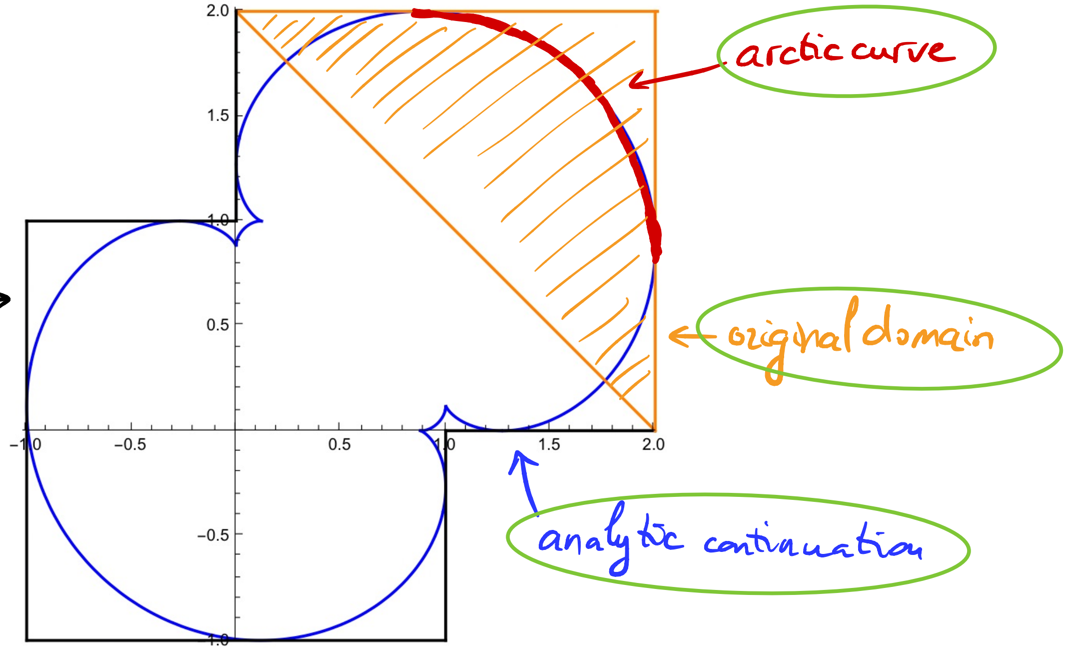


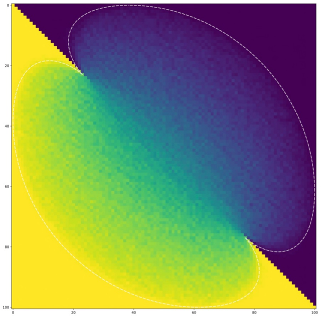
F_1
F_0
F_3
F_5
F_4
L
F_2
2.0
1.5
1.0
0.5
−2.0
−1.5
−1.0
−0.5

OPEN QUESTION: FIND A DOMINO TILING IN BIJECTION WITH $\widetilde{20V}_3$?

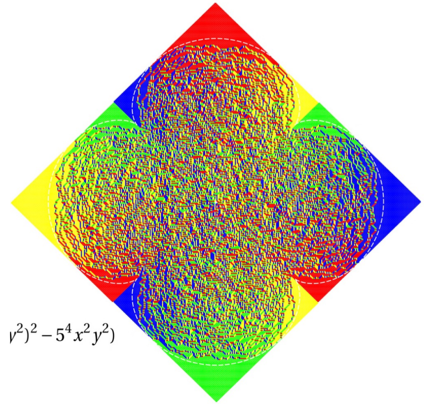
Hint: arctic curve for the uniform counting (by tangent method)

domino tiling ?





МЕРСИ!



Refs:

- Di Francesco, Guitter Elec. Journ. Comb. 27 No 2 (2020) P2.13
- Debin, Di Francesco, Guitter Jour. Stat. Phys. 179 (2020) pp 33–89
- Di Francesco, ArXiv 2102.02920 [math.CO] (2021)
- Di Francesco, ArXiv 2106.02098 [math-ph] (2021)
- P. Di Francesco, in preparation (2022)